ECHNOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 64 NUMBER 2

CONTENTS

Trammel Rotors in Regular Polygons . . . MICHAEL GOLDBERG 71
A Method for Deriving Numerical Differentiation Formulas . . . . . R. T. GREGORY 79
A Curious Trigonometric Identity . . . . . R. M. ROBINSON 83
Mathematical Notes . . . . . G. S. SMITH, K. F. MOPPERT, ... M. P. DRAZIN, M. S. KLAMKIN, VICTOR THÉBAULT, ... R. C. KAO AND L. H. ZETTERBERG, KURT BING 86
Classroom Notes. . . A. OPPENHEIM, D. H. LEHMER, L. R. FORD 101
Elementary Problems and Solutions 109
Advanced Problems and Solutions 116
Recent Publications 126
News and Notices 131
The Mathematical Association of America 137
Itineraries of Visiting Lecturers, 1956–57 137
New Members. 139
October Meeting of the Minnesota Section 140
Calendar of Future Meetings 142

FEBRUARY 1957

PUBLIC LIBRARY
FEB 26 1957
DETROIT

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Ralph D. James, *Editor*

ASSOCIATE EDITORS

RICHARD V. ANDREE
HOWARD EVES
G. E. HAY
L. M. KELLY
IVAN NIVEN
C. O. OAKLEY
J. M. H. OLMSTED
EDITH R. SCHNECKENBURGER
E. P. STARKE
H. S. ZUCKERMAN

EDITORIAL CORRESPONDENCE should be addressed to the Editor, R. D. James, Department of Mathematics, University of British Columbia, Vancouver 8, Canada. Articles should be typewritten and double-spaced. The greatest possible care should be taken in preparing the manuscript.

ADVERTISING CORRESPONDENCE should be addressed to F. R. Olson, University of Buffalo, Buffalo 14, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.
(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, G. B. Price, University of Kansas
First Vice-President, R. V. Churchill, University of Michigan
Second Vice-President, B. W. Jones, University of Colorado
Secretary-Treasurer, H. M. Gehman, University of Buffalo
Associate Secretary, Edith R. Schneckenburger, University of Buffalo
Editor, R. D. James, University of British Columbia

Additional Members of the Board of Governors: H. M. Bacon, M. A. Basoco, H. W. Brinkmann, P. H. Daus, H. H. Downing, W. L. Duren, Jr., P. D. Edwards, N. J. Fine, R. M. Foster, F. F. Helton, M. S. Hendrickson, A. S. Householder, R. C. Huffer, C. A. Hutchinson, L. W. Johnson, G. K. Kalisch, E. C. Kiefer, F. W. Kokomoor, Z. L. Loflin, L. L. Lowenstein, Saunders MacLane, E. J. McShane, J. R. Mayor, A. E. Meder, Jr., W. H. Myers, Ivan Niven, Morris Ostrofsky, O. J. Ramler, J. F. Randolph, C. B. Read, C. R. Sherer, M. F. Smiley, B. M. Stewart, G. B. Thomas, Jr., Bernard Vinograde, M. A. Zorn

---

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $5.00. For non-members the subscription price is $6.00.

Published by the Association at Menasha, Wisconsin, and Buffalo, N. Y., during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

TRAMMEL ROTORS IN REGULAR POLYGONS

MICHAEL GOLDBERG, Bureau of Ordnance, U.S. Navy

1. Introduction. A rotor in a polygon (plane or spherical) is defined as a closed convex curve which remains tangent to all the sides of the polygon during a complete rotation of the curve. All the plane rotors were obtained and described analytically by Meissner [1] by means of a Fourier series expansion. Extensions to spherical rotors were shown by the author [3, 4].

In "*Circular-arc Rotors in Regular Polygons*" [2], the author derived kinematically a rotor bounded entirely by arcs of circles for each regular polygon. In "*Rotors in Spherical Polygons*" [3], he obtained an analogous rotor for each regular spherical polygon. For odd polygons, this rotor had two axes of symmetry; for even polygons ($n>4$), only one axis of symmetry.

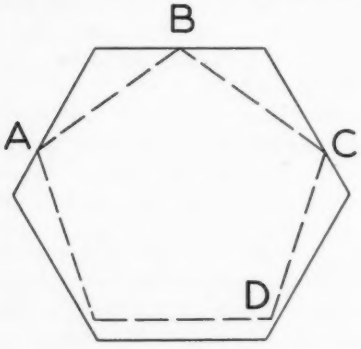
In the present paper, a new series of circular-arc rotors is derived in the plane characterized by the possession of all the higher orders of symmetry. The corresponding rotors in spherical polygons are derived in a similar manner. However, the bounding arcs of spherical rotors are not all circular. In the plane, the generation is accomplished by the motion of vertices or other points of the rotor along straight lines. On the sphere, rotor points move along great circles. Hence, to distinguish the circular-arc rotors and the corresponding spherical rotors from the more general ones, I choose to call them *trammel rotors*.

The trammel rotors in the square include the Reuleaux polygons which are rotors bounded by arcs whose radii are equal to the edge of the square. The trammel rotors in the triangle include rotors whose radii are all equal to the height of the triangle. These were investigated by Fujiwara [7]. However, for the pentagon, there are no trammel rotors which have the same radii; at least two different radii are needed. Among those which have the height of the pentagon as one radius are the regular rotor described by Fujiwara [7, p. 245], shown in Figure 3, the biaxial rotor described by the author [2] and the family described here.

2. Construction of the (n-1)-lobed regular trammel rotor in the n-gon. Given a regular n-gon, inscribe a regular $(n-1)$-gon in the following manner. See Figure 1. Take the midpoint B of a side of the given n-gon as a vertex of the $(n-1)$-gon. Draw lines from this point inclined at the angle $\pi/(n-1)$ with the side. The intersections A and C of these drawn lines with the adjacent sides of the n-gon will be taken as the adjacent vertices of the $(n-1)$-gon. With the sides AB and BC as a beginning, complete the $(n-1)$-gon.

Now rotate the $(n-1)$-gon counter-clockwise within the n-gon while the vertices B and C are constrained to move along the sides of the n-gon. After a rotation of $\alpha=2\pi/(n-1)-2\pi/n=2\pi/n(n-1)$, the next vertex D will touch a side of the n-gon. The relative positions of the polygons at mid-position of the motion are shown in Figure 2. The envelopes of the positions of the fixed sides of the n-gon on the moving plane of the $(n-1)$-gon are $n-2$ arcs of circles by Bobillier's theorem [2, pp. 393–394]. The centers of these circles will lie on a

circle through B and C and the intermediate vertex of the n-gon. The totality of these points will be equally spaced on the circle. Also, these centers lie on diagonals of the n-gon. Hence, the radii of the circular arcs are the distances of these diagonals from the sides of the n-gon parallel to them.



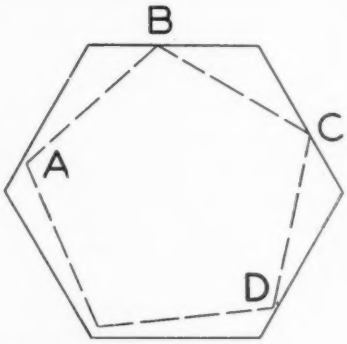


FIG. 1 and FIG. 2. Vertices of regular $(n-1)$-lobed rotor in n-gon.

The rotation of the $(n-1)$-gon is now continued but with the vertices C and D moving along the sides of the n-gon. Each time a new vertex touches a side of the n-gon, it continues by moving along this side. The contour of the rotor is thus molded by the sides of the n-gon. The complete contour is formed by a rotation of $[(n-1)/2]\alpha$ where $[(n-1)/2]$ is the integral part of $(n-1)/2$. Further rotation will retrace the same contour. The deflection angle of the contour at a cusp (or vertex) of the rotor is also α.

3. Construction of the $(n+1)$-lobed regular trammel rotor in the n-gon. Given a regular n-gon, inscribe a regular $(n+1)$-gon as the $(n-1)$-gon was inscribed in Section 2. Draw lines from the midpoint B of a side making angles $\pi/(n+1)$ with the side. The intersections A and C with the adjacent sides of the n-gon will be taken as adjacent vertices of the $(n+1)$-gon. With the sides AB and BC as a beginning, complete the $(n+1)$-gon.

Now rotate the $(n+1)$-gon within the n-gon while the vertices B and C are constrained to move along the sides of the n-gon. After a rotation of $\alpha=2\pi/n-2\pi/(n+1)=2\pi/n(n+1)$, a new vertex will touch a side of the n-gon. Continue the rotation as in Section 2 by keeping two vertices moving along sides of the n-gon. The contour of the rotor is thus molded by the sides of the n-gon. The radii are the same as in Section 2, and the total measure of the arcs of each radius is the same.

4. The regular trammel rotors. A completed regular four-lobed rotor in the regular pentagon is shown in Figure 3. The circle of one set of centers is shown as well as the radii of the sectors. The normals at the points of contact intersect in the instantaneous center of rotation.

The regular $(n-1)$-lobed and $(n+1)$-lobed rotors for $n=3$, 4, 5, and 6 that

are generated by the methods of Sections 2 and 3 are shown in Figure 4 for comparison. These should be compared also with the previously published set of trammel rotors [2, Fig. 4]. The duals of these rotors are another set of regular trammel rotors. The rotors in the even polygons are self-dual.

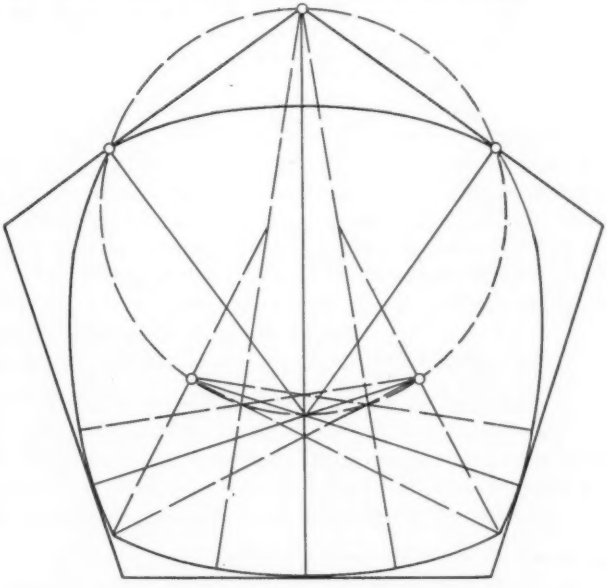


FIG. 3. Regular four-lobed rotor in regular pentagon.

5. Other trammel rotors. Every rotor in a regular n-gon is described by the polar tangential equation, due to Meissner [1],

$$p(\theta) = a_0 + \sum_{k=1}^{\infty} a_k \sin k\theta + \sum_{k=1}^{\infty} b_k \cos k\theta, \tag{1}$$

where

$$a_k = b_k = 0 \quad \text{for} \quad k \not\equiv \pm 1 \pmod{n}.$$

The weighted mean of the equations for two rotors in an n-gon will give another equation satisfying the same condition on the coefficients. Therefore, the new equation will also describe a rotor in the same n-gon. The radius of curvature ρ is given by the equation

$$\rho = p(\theta) + p''(\theta) \geqq 0. \tag{2}$$

For convexity, ρ cannot be negative. This condition imposes a limiting relation among the constants a_k and b_k.

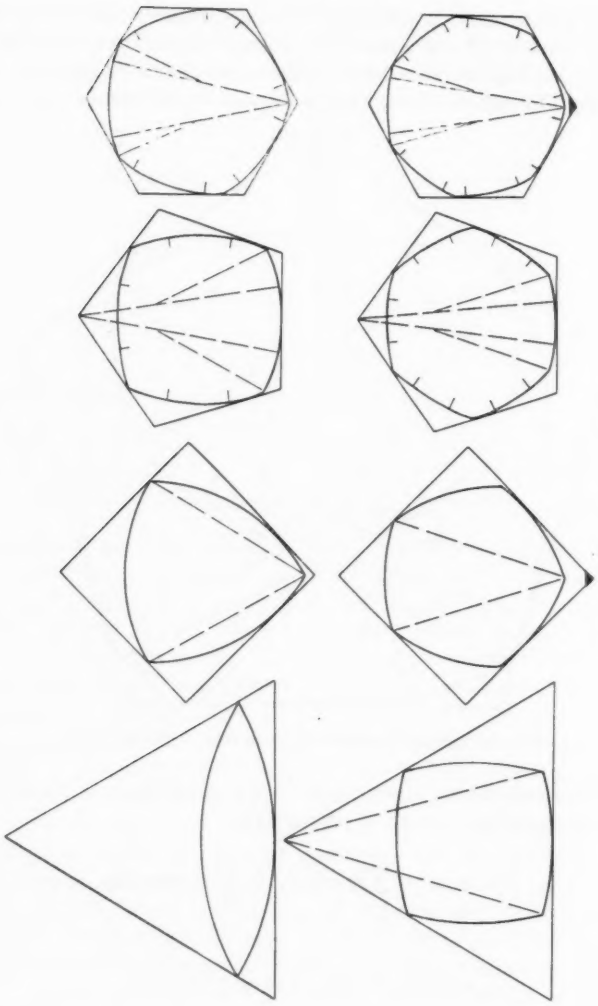


FIG. 4. Regular trammel rotors.

Since a rotor in a kn-gon is also a rotor in an n-gon, the foregoing sections show, for each regular polygon, the constructions for an infinity of different trammel rotors. They are not, however, all the possible rotors. New ones are described by linear combinations of the equations for those previously derived.

For example, if two trammel rotors in a polygon are described by the equations

$$p_1 = f_1(\theta) \quad \text{and} \quad p_2 = f_2(\theta),$$

then new trammel rotors in the same polygon are their weighted means given analytically by the equation

$$p_3 = \frac{uf_1(\theta) + vf_2(\theta)}{u + v},$$

where u and v are any real numbers. This equation includes non-convex as well as convex curves. Of course, only the convex curves may be used. If one of the given rotors is a circle, then the new rotors are similar to the parallel rotors. The rotor similar to the convex internal parallel rotors are the *dual rotors*.

These new rotors are obtained geometrically as follows. Each arc of a new rotor has a center which is the weighted mean of the two centers of the corresponding arcs of the two given rotors and whose radius is the weighted mean of the corresponding arcs of the given rotors.

The foregoing may be summarized in the following theorems.

THEOREM 1. *A regular $(n-1)$-lobed rotor in a regular n-gon can be made of $n-1$ equal parts, each composed of a circular arc of measure $\alpha = 2\pi/n(n-1)$ and radius r_1 equal to the height of the n-gon, flanked on each side by successively tangent circular arcs of measure α and radii $r_2, r_3, \cdots, r_k$, where $k = [(n-1)/2]$ and $r_2, r_3, \cdots, r_k$ are the distances of the diagonals of the n-gon from the parallel sides.*

THEOREM 2. *A regular $(n+1)$-lobed rotor in a regular n-gon can be made of $n+1$ equal parts, each composed of a circular arc of measure $\alpha = 2\pi/n(n+1)$ and radius r_1 equal to the height of the n-gon, flanked on each side by successively tangent circular arcs of measure α and radii $r_2, r_3, \cdots, r_k$, where $k = [(n-1)/2]$, and $r_2, r_3, \cdots, r_k$ are the distances of the diagonals of the n-gon from the parallel sides.*

THEOREM 3. *Parallel curves of the rotors of Theorems 1 and 2 are also rotors in regular n-gons. The cusps of the rotors in Theorems 1 and 2 are replaced by circular arcs of measure α.*

THEOREM 4. *A convex linear combination of the trammel rotors of Theorems 1, 2, and 3 in an n-gon is also a trammel rotor in the n-gon.*

The rotors described in the foregoing paragraphs still do not exhaust the possible trammel rotors. For example, the rotors of four and five arcs for the equilateral triangle, described by Fujiwara [7], are not derivable by finite combinations in this way. However, both trammel rotors and non-trammel rotors may conceivably be represented asymptotically by a linear combination of an infinite series of $(n-1)$-lobed and $(n+1)$-lobed regular trammel rotors. It would be interesting to find a minimal set of trammel rotors from which each of the others could be obtained by a finite number of linear combinations.

6. On rotors of minimum area. All the rotors in a given polygon have the same perimeter. However, the areas vary. In each case, the maximum area is attained by the circle. The minimum area and the shapes that attain it are not generally known. In fact, only for the equilateral triangle and the square have minimum rotors been determined. The minimum rotor for the triangle is bounded by two circular arcs whose radius is equal to the height of the triangle. The minimum rotor for the square is bounded by three equal circular arcs whose radius is equal to the height of the square.

Fujiwara, who used an analytical method for deriving the minimum rotors for the triangle and the square [6, 7], stated that his method was not successful for the pentagon [7, p. 246]. However, he submitted the regular trammel rotor (described in Section 2 and pictured in Figure 3) as a possible contender for this distinction.

A family of four-lobed trammel rotors in the pentagon is constructable by re-arranging the arcs of the Fujiwara rotor. If e is the edge of the pentagon, then the radii of the arcs are given by the equations

$$r_1 = e(1 + 2\cos 72°) = \text{height of pentagon},$$

$$r_2 = e \sin 72°.$$

The bounding arcs of a quadrant of a rotor of the family are given by arcs of the following radii and angles

$$(r_1, 18° - \beta), \quad (r_2, 2\beta), \quad (0, 36°), \quad (r_2, 36° - 2\beta), \quad (r_1, \beta),$$

where $0 \leqq \beta \leqq 9°$. When $\beta = 9°$, the area is minimized and the Fujiwara rotor is obtained. When $\beta = 0$, the four-arc rotor of biaxial symmetry is obtained. These rotors may be derived as dual rotors of the mean of two biaxial four-arc rotors placed at an angle to each other. When the angle is 90°, the Fujiwara rotor is obtained.

The area A of a closed convex curve is given by the equation

$$A = \frac{1}{2}\int_0^{2\pi} \rho p d\theta. \tag{3}$$

The determination of the rotor of minimum area may be formulated as the determination of the coefficients a_k and b_k which satisfy equations (1) and (2) and which minimize A in equation (3).

An examination of the possibilities has led the author to venture the following conjecture.

Among all the possible rotors in a given n-gon, the rotor of least area is the regular trammel rotor of $n-1$ lobes.

Rotors may be ranked by the quantity $K = A/L^2$ where A is the area and L is the perimeter. Then K for several rotors in the pentagon are listed below.

Description of rotor in pentagon	K
1. Regular trammel rotor (Fujiwara rotor)	.07603
2. Trammel rotor with biaxial symmetry	.07679
3. Basic rotor $p=a(15+\cos 4\theta)$ [4]	.07692
4. Dual rotor of 2	.07724
5. Circle	.07958

7. Regular trammel rotors in spherical polygons. The same procedure for generating trammel rotors in the plane can be used for generating the regular

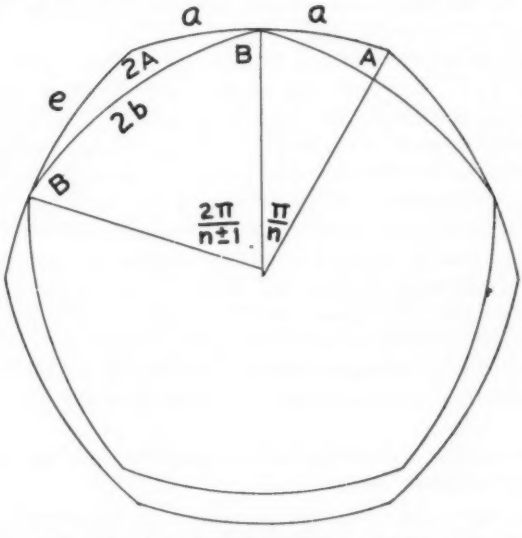


FIG. 5. Vertices of regular rotor in spherical polygon.

trammel rotors in spherical polygons. See Figure 5. If the side of the given n-gon is $2a$, and the side of the inscribed $(n\pm1)$-gon is $2b$, the relation between them is obtained by eliminating B, C and e from the following equations.

$$\sin A \cos a = \cos \pi/n,$$

$$\sin B \cos b = \cos \pi/(n \pm 1),$$

$$\sin e \sin 2A = \sin 2b \cos B,$$

$$\cos e = \cos a \cos 2b + \sin a \sin 2b \sin B.$$

If two consecutive vertices of the inscribed polygon are moved along adjacent sides of the fixed n-gon until another vertex touches the n-gon, the envelopes of the other sides of the n-gon on the surface of the moving inscribed polygon will be part of the boundary of the generated rotor. Similar motions will complete the generation of the rotor boundary. The 90° parallels will give a set of

regular trammel rotors external to the polar polygons.

The boundary curves are not plane and, therefore, they are not circles. However, they do have the property, shared by the circle in the plane, of being the locus of a vertex of a moving constant angle whose sides pass through two fixed points.

Other spherical trammel rotors, possessing only one and two axes of symmetry, were described in an earlier paper by the author [3].

A law of combination for trammel rotors on the sphere, similar to the law of combination described in Section 5, has not yet been obtained. However, a law of combination for the basic rotors, which resemble the regular trammel rotors, has been derived by the author [4].

8. Trammel rotors in polyhedra. Surfaces of constant width may be considered as rotors in a cube. Trammel rotors in a polyhedron may be taken as those generated by restraining the motion of each of three of its points to a plane for each finite interval of the motion. The surfaces generated are spherical and toroidal surfaces. One obvious family of such rotors in a cube is the family of surfaces of revolution obtained by rotating a symmetrical trammel rotor in a square about its axis of symmetry. Other trammel rotors are also possible. Meissner described one [8] which has the four vertices of a regular tetrahedron, four spherical surfaces opposite them and three toroidal strips along three of the edges of the tetrahedron. Still others may be based on any tetrahedron and will have, in general, eight spherical surfaces, (two centered on each vertex), and six toroidal surfaces (one corresponding to each edge of the tetrahedron). As in the case of plane rotors, new rotors may be obtained as combinations of known rotors.

Rotors in the regular tetrahedron and octahedron exist, but it is not known if any of these are trammel rotors.

References

1. E. Meissner, Über die Anwendung von Fourier-Reihen auf einige Aufgaben der Geometrie und Kinematik, Vierteljschr. Naturforsch. Ges. Zürich, vol. 54, 1909, pp. 309–329.

2. M. Goldberg, Circular-arc rotors in regular polygons, this MONTHLY, vol. 55, 1948, pp. 393–402.

3. ———, Rotors in spherical polygons, J. Math. Physics, vol. 30, 1952, pp. 235–244.

4. ———, Basic rotors in spherical polygons, J. Math. Phys., vol. 34, 1956, pp. 322–327.

5. W. Blaschke, Konvexe Bereiche gegebener Breite und kleinsten Inhalts, Math. Ann., vol. 76, 1914, pp. 504–513.

6. M. Fujiwara and S. Kakeya, On some problems of maxima and minima for the curve of constant breadth and the in-revolvable curve of the equilateral triangle, Tôhoku Math. J., vol. 11, 1917, pp. 92–110.

7. M. Fujiwara, Über die innen-umdrehbare Kurve eines Vieleckes, Sci. Rep., Tôhoku Univ., Ser. 1, vol. 8, 1919, pp. 221–246.

8. E. Meissner, Drei Gipsmodelle von Flächen konstanter Breite, Z. Math. Phys., vol. 60, 1912, pp. 92–94.

9. W. Wunderlich, Über eine Klasse zwangläufiger höherer Elementenpaare, Z. Angew. Math. Mech., vol. 19, 1939, pp. 177–181. (This paper describes examples of non-convex trammel rotors in star polygons.)

A METHOD FOR DERIVING NUMERICAL DIFFERENTIATION FORMULAS

R. T. GREGORY, University of California, Goleta, and Computer Control Company

The problem of polynomial interpolation usually involves the approximation of a function $y=y(x)$ by a suitably chosen nth degree polynomial $P_n(x)$ such that $y_i \equiv y(x_i) = P_n(x_i)$ for $i=0, 1, \cdots, n$. It is well known that a unique polynomial of degree n passes through $n+1$ distinct points (x_0, y_0), (x_1, y_1), $\cdots$, (x_n, y_n), where $x_0<x_1< \cdots <x_n$, though it may be derived in different ways and hence may appear in different forms (*e.g.*, the interpolation formulas of Newton, Lagrange, *etc.*). We can obtain an approximation to $y(x_k)$, if $x_k \neq x_i$, by computing $P_n(x_k)$. If $x_0<x_k<x_n$ this process is called interpolation; otherwise extrapolation.

To find approximations to the derivatives of $y(x)$ we usually differentiate the interpolating polynomial $P_n(x)$. However, in this discussion we shall be interested only in values of the derivatives at the mesh points x_i and in this particular case there is no need to find $P_n(x)$. Instead we propose to find formulas of the form

$$y_i^{(k)} = h^{-k} \sum_{j=0}^{n} A_{ij}^{ks} y_j + O(h^s), \tag{1}$$

where $y_i^{(k)}$ denotes the kth derivative of $y(x)$ evaluated at the point x_i and $O(h^s)$ indicates the truncation error. Thus, in effect, we can express the derivatives at the mesh points as linear combinations of the known ordinates y_i. For examples of such formulas see Milne [1], Hildebrand [2], and Nielsen [3].

The purpose of this paper is to describe an algorithm for finding the coefficients A_{ij}^{ks} for a formula of type (1) when the order of the derivative k and the quantity s are specified. We shall use an "undetermined coefficients" method which requires that $y(x)$ possess a continuous $(k+s)$th derivative in the region of consideration.

Consider the Taylor expansion of y_j about the point x_i

$$y_j = \sum_{t=0}^{k+s-1} (j-i)^t \frac{h^t}{t!} y_i^{(t)} + \frac{(j-i)^{k+s} h^{k+s}}{(k+s)!} y^{(k+s)}(\xi_{ij}), \tag{2}$$

where ξ_{ij} lies in (x_i, x_j). If we write the expansion (2) for $j=0, 1, 2, \cdots, n$ and then form a linear combination of these ordinates (where the coefficients are denoted by A_{ij}^{ks}) we obtain

$$\sum_{j=0}^{n} A_{ij}^{ks} y_j = y_i \sum_{j=0}^{n} A_{ij}^{ks} + h y_i^{(1)} \sum_{j=0}^{n} (j-i) A_{ij}^{ks} + \cdots + \frac{h^{k+s-1}}{(k+s-1)!} y_i^{(k+s-1)} \sum_{j=0}^{n} (j-i)^{k+s-1} A_{ij}^{ks} + \frac{h^{k+s}}{(k+s)!} \sum_{j=0}^{n} (j-i)^{k+s} A_{ij}^{ks} y^{(k+s)}(\xi_{ij}),$$

$$= \sum_{m=0}^{k+s-1} \frac{h^m}{m!} y_i^{(m)} B_{im}^{ks} + \frac{h^{k+s}}{(k+s)!} \sum_{j=0}^{n} (j-i)^{k+s} A_{ij}^{ks} y^{(k+s)}(\xi_{ij}), \tag{3}$$

where

$$B_{im}^{ks} = \sum_{j=0}^{n} (j-i)^m A_{ij}^{ks}. \tag{4}$$

Dividing (3) by h^k gives

$$h^{-k} \sum_{j=0}^{n} A_{ij}^{ks} y_j = \sum_{m=0}^{k+s-1} \frac{h^{m-k}}{m!} y_i^{(m)} B_{im}^{ks} + E_i^{ks}, \tag{5}$$

where

$$E_i^{ks} = \frac{h^s}{(k+s)!} \sum_{j=0}^{n} (j-i)^{k+s} A_{ij}^{ks} y^{(k+s)}(\xi_{ij}), \tag{6}$$

and we determine the coefficients A_{ij}^{ks} so as to make

$$B_{im}^{ks} = \delta_{mk} k! \qquad (m = 0, 1, \cdots, k+s-1). \tag{7}$$

Using (7) in (5) we obtain

$$y_i^{(k)} = h^{-k} \sum_{j=0}^{n} A_{ij}^{ks} y_j - E_i^{ks}, \tag{8}$$

where (6) shows (8) to be a formula of type (1).

Let us use (4) to write the system of linear equations (7) in the form

$$\begin{bmatrix} 1 & 1 & 1 & \cdot & 1 \\ (-i) & (1-i) & (2-i) & \cdot & (n-i) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (-i)^k & (1-i)^k & (2-i)^k & \cdot & (n-i)^k \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (-i)^t & (1-i)^t & (2-i)^t & \cdot & (n-i)^t \end{bmatrix} \begin{bmatrix} A_{i0}^{ks} \\ A_{i1}^{ks} \\ A_{i2}^{ks} \\ \cdot \\ \cdot \\ A_{in}^{ks} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ k! \\ \cdot \\ 0 \end{bmatrix}, \tag{7'}$$

where $t = k+s-1$.

There are $k+s$ equations in the $n+1$ variables A_{ij}^{kn} and in general we must have

$$n+1 \geqq k+s. \tag{9}$$

if the equations are to be consistent. The inequality (9) gives us a clear insight into the relationship that exists among the quantities n, k, and s. For instance, if we are given k and s we automatically have a lower bound on the number of ordinates y_i necessary for a formula of type (1). On the other hand, if we have a fixed value of n, and k is given, we have an upper bound on the accuracy of the formula of type (1) which can be derived.

In the usual case where $n+1=k+s$ we have a unique solution to the system (7′) since the coefficient matrix is square and non-singular. The non-singularity follows directly if we observe that the determinant has the form of the determinant of Vandermonde

$$\begin{vmatrix} 1 & 1 & 1 & \cdot & 1 \\ z_0 & z_1 & z_2 & \cdot & z_n \\ z_0^2 & z_1^2 & z_2^2 & \cdot & z_n^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z_0^n & z_1^n & z_2^n & \cdot & z_n^n \end{vmatrix} = \prod_{j,k=0,j>k}^{n} (z_j - z_k)$$

and, in (7′), $z_j - z_k \neq 0$ for $j \neq k$.

The writer has found occasion to choose $n+1>k+s$, *i.e.*, more ordinates were used than absolutely necessary to obtain a differentiation formula for a specified k and s. In this case it is easily seen that the system (7′) has rank $k+s$ and hence the solutions A_{ij}^{kn} are functions of $(n+1)-(k+s)$ arbitrary parameters.

To illustrate the algorithm, let us find formulas for the second derivative which have a truncation error $O(h^3)$. Here $k=2$ and $s=3$. First we set $n+1=5$ (*i.e.*, we use 5 mesh points) and then find the unique set of coefficients for the cases $i=0, 1, 2, 3, 4$. The algorithm merely involves solving (7′) for these five values of i. For instance, when $i=0$ we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 9 & 16 \\ 0 & 1 & 8 & 27 & 64 \\ 0 & 1 & 16 & 81 & 256 \end{bmatrix} \begin{bmatrix} A_{00}^{23} \\ A_{01}^{23} \\ A_{02}^{23} \\ A_{03}^{23} \\ A_{04}^{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix},$$

with the solutions

$$A_{00}^{23} = \frac{35}{12}, \quad A_{01}^{23} = -\frac{104}{12}, \quad A_{02}^{23} = \frac{114}{12}, \quad A_{03}^{23} = -\frac{56}{12}, \quad A_{04}^{23} = \frac{11}{12}.$$

In like manner the other four sets of coefficients can be found and we have the following five formulas:

$$y_0^{(2)} = \frac{1}{12h^2}[35y_0 - 104y_1 + 114y_2 - 56y_3 + 11y_4],$$

$$y_1^{(2)} = \frac{1}{12h^2}[11y_0 - 20y_1 + 6y_2 + 4y_3 - y_4],$$

$$y_2^{(2)} = \frac{1}{12h^2}[-y_0 + 16y_1 - 30y_2 + 16y_3 - y_4],$$

$$y_3^{(2)} = \frac{1}{12h^2}[-y_0 + 4y_1 + 6y_2 - 20y_3 + 11y_4],$$

and

$$y_4^{(2)} = \frac{1}{12h^2}[11y_0 - 56y_1 + 114y_2 - 104y_3 + 35y_4].$$

The errors in these formulas are given by (6). For example, the error in $y_2^{(2)}$ becomes

$$E_2^{23} = \frac{h^5}{90}[2y^{(5)}(\xi_{20}) - y^{(5)}(\xi_{21}) + y^{(5)}(\xi_{23}) - 2y^{(5)}(\xi_{24})],$$

and

$$|E_2^{23}| \leqq \frac{h^5}{15}M,$$

where M is the maximum absolute value of $y^{(5)}(x)$ in the interval (x_0, x_4).

Notice that the coefficient matrix on the left in equation (7′) is a function of the combination $k+s$. Thus we use the same matrix for finding a formula for $y_i^{(k-r)}$ with truncation error $O(h^{s+r})$ that we use for finding a formula for $y_i^{(k)}$ with truncation error $O(h^s)$. The only difference in the two cases is the vector on the right.

References

1. W. E. Milne, Numerical Calculus, Princeton, 1949, pp. 96–99.
2. F. B. Hildebrand, Introduction to Numerical Analysis, New York, 1956, p. 82.
3. K. L. Nielsen, Methods in Numerical Analysis, New York, 1956, pp. 342–347.

A CURIOUS TRIGONOMETRIC IDENTITY

R. M. ROBINSON, University of California, Berkeley

Recently I discovered, to my surprise, that

$$|\sin(x+iy)| = |\sin x + \sin iy|$$

for real values of x and y. This is easily verified. In the first place, by the addition formula, we have

$$\begin{aligned}\sin(x+iy) &= \sin x \cos iy + \cos x \sin iy \\ &= \sin x \cosh y + i \cos x \sinh y.\end{aligned}$$

From this, it follows that

$$\begin{aligned}|\sin(x+iy)|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &= \sin^2 x (1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y \\ &= \sin^2 x + \sinh^2 y = |\sin x + i \sinh y|^2 \\ &= |\sin x + \sin iy|^2,\end{aligned}$$

which yields the desired identity.

In a similar way, or using the above identity, we find also that

$$|\sinh(x+iy)| = |\sinh x + \sinh iy|.$$

We shall now show that z, $\sin z$, and $\sinh z$ are essentially the only functions with this property.*

THEOREM. *If $f(z)$ is regular for $|z|<r$, and satisfies the functional equation*

$$|f(x+iy)| = |f(x)+f(iy)|$$

for real values of x and y, then

$$f(z) = Az, \quad f(z) = A\sin bz, \quad \text{or} \quad f(z) = A\sinh bz,$$

where A and b are constants, and b is real.

Proof. From the preceding, it is clear that these functions do satisfy the functional equation. It remains to show that there is no other solution.

Suppose that $f(z)$ is regular for $|z|<r$, and satisfies the functional equation. Putting $z=0$, we find that $f(0)=0$. Hence we may put

* After this result was presented to the Northern California Section of the Association on January 14, 1956, Professor G. Szegö called my attention to a related paper by Einar Hille, *A Pythagorean functional equation*, Ann. of Math., vol. 24, 1922, pp. 175–180. Hille draws the same conclusion for a different functional equation, namely, $|f(x+iy)|^2 = |f(x)|^2 + |f(iy)|^2$. It is not clear how to modify Hille's method to apply to the functional equation considered here. On the other hand, the calculations of the present paper need only minor modifications to yield Hille's result.

$$f(z) = \sum_{n=p}^{\infty} c_n z^n \qquad \text{for } |z| < r,$$

where $p \geqq 1$. Unless $f(z)$ vanishes identically, we may suppose that $c_p \neq 0$. It will clearly be sufficient to consider normalized solutions, with $c_p = 1$.

We see that

$$|f(x+iy)|^2 = \sum_k c_k(x+iy)^k \cdot \sum_l \bar{c}_l(x-iy)^l$$
$$= \sum_{k,l} c_k\bar{c}_l(x+iy)^k(x-iy)^l$$

and

$$|f(x)+f(iy)|^2 = \sum_k c_k[x^k+(iy)^k] \cdot \sum_l \bar{c}_l[x^l+(-iy)^l]$$
$$= \sum_{k,l} c_k\bar{c}_l[x^k+(iy)^k][x^l+(-iy)^l].$$

If the given functional equation is to be satisfied, then the terms in these two series of degree m in x and y must agree. This yields the recursion formula

$$\sum_{k+l=m} c_k\bar{c}_l(x+iy)^k(x-iy)^l = \sum_{k+l=m} c_k\bar{c}_l[x^k+(iy)^k][x^l+(-iy)^l]$$

for the coefficients of $f(z)$.

The first case of this recursion formula to be considered is $m = 2p$. This yields

$$(x+iy)^p(x-iy)^p = [x^p+(iy)^p][x^p+(-iy)^p],$$

or

$$(x^2+y^2)^p = \begin{cases} x^{2p}+y^{2p} & \text{if } p = 2q+1, \\ (x^p+y^p)^2 & \text{if } p = 4q, \\ (x^p-y^p)^2 & \text{if } p = 4q+2. \end{cases}$$

It is seen that this is an identity only if $p = 1$. Thus the expansion of $f(z)$ must necessarily start with the first power of z, so that

$$f(z) = z + c_2z^2 + c_3z^3 + \cdots.$$

Since the terms in the recursion formula occur in conjugate pairs, it may be rewritten in the form

$$\Re \sum_{1 \leq l \leq m/2}{}' c_{m-l}\bar{c}_l\{(x+iy)^{m-l}(x-iy)^l - [x^{m-l}+(iy)^{m-l}][x^l+(-iy)^l]\} = 0,$$

where the prime on the summation sign indicates that for even m, the term with $l = m/2$ must be multiplied by $1/2$. Putting $m = 3$, we find that

$$\Re\{c_2(x-iy)[(x+iy)^2-(x^2-y^2)]\} = 0$$

or

$$\Re\{c_2(x - iy) \cdot 2ixy\} = 0.$$

Thus both the real and imaginary parts of c_2 must vanish, so that $c_2 = 0$. Next, putting $m = 4$, we find that

$$\Re\{c_3(x - iy)[(x + iy)^3 - (x^3 - iy^3)]\} = 0$$

or

$$\Re\{c_3(x^2 + y^2) \cdot 3ixy\} = 0.$$

Thus the imaginary part of c_3 must vanish, or c_3 must be real. Finally, putting $m = n+1$, where $n > 3$, we find that

$$\Re\{c_n(x - iy)[(x + iy)^n - (x^n + i^n y^n)] + \cdots\} = 0,$$

where the omitted terms involve earlier coefficients only. From this we have

$$\Re\{c_n(x - iy)[nx^{n-1}iy + (1/2)n(n-1)x^{n-2}(iy)^2 + \cdots] + \cdots\} = 0.$$

and hence

$$\Re\{c_n[inx^n y - (1/2)n(n-3)x^{n-1}y^2 + \cdots] + \cdots\} = 0.$$

This shows that both the real and imaginary parts of c_n are determined uniquely in terms of earlier coefficients for $n > 3$, so that c_n itself is so determined. Summarizing, we have

$$f(z) = z + c_3 z^3 + c_4 z^4 + \cdots,$$

where c_3 is real, and the remaining coefficients are uniquely determined in terms of c_3.

On the other hand, the known solutions

$$z, \qquad (1/b)\sin bz, \qquad (1/b)\sinh bz$$

provide solutions of this form with $c_3 = 0$, $c_3 = -b^2/6$, and $c_3 = b^2/6$, that is, solutions with arbitrary real c_3. Since the remaining coefficients are uniquely determined in terms of c_3, there can be no other normalized solutions. Every solution is found from one of these by multiplying by a constant factor.

AN ACKNOWLEDGEMENT

F. A. VALENTINE

Theorem 1 in my paper "*The motion of a particle constrained to move on a rough convex curve*," this MONTHLY, vol. 63, 1956, pp. 16–20, is essentially contained on page 221 of "*Dynamics of a particle*" by Tait and Steele, 5th Edition, Macmillan, 1882.

MATHEMATICAL NOTES

EDITED BY IVAN NIVEN, University of Oregon

Material for this department should be sent to Ivan Niven, Department of Mathematics, University of Oregon, Eugene, Oregon.

EXPRESSION OF IRRATIONALS OF ANY DEGREE AS REGULAR CONTINUED FRACTIONS WITH INTEGRAL COMPONENTS

G. S. SMITH, Northwood, Middlesex, England

By means of Thiele's Theorem [1], $f(x+h)$ can be expressed as a continued fraction in terms of h, $f(x)$, and the reciprocal derivatives of $f(x)$, thus

$$f(x+h) = f(x) + \cfrac{h}{rf(x) + \cfrac{h}{2rrf(x) + \cfrac{h}{3rr_2f(x) + \cdots}}}, \tag{1}$$

where

$$rf(x) = \frac{1}{f'(x)}, \qquad r_2f(x) = f(x) + 2rrf(x),$$

and in general

$$r_nf(x) = r_{n-2}f(x) + nrr_{n-1}f(x).$$

The purpose of this note is to show that (1) can be applied to the problem of expressing the mth root of the integral number N as a continued fraction with a simple and regular development.

We will write N in the form $\alpha^m+\beta$, where α, β, and m are all integral, and apply (1) with $f(x)=x^{1/m}$, finally giving h and x the values β and α^m respectively.

We have

$$a_0 = x^{1/m},$$

$$a_1 = rx^{1/m} = mx^{(m-1)/m},$$

$$a_2 = 2rrx^{1/m} = \frac{2x^{1/m}}{m-1}, \qquad r_2x^{1/m} = \frac{m+1}{m-1}x^{1/m},$$

$$a_3 = 3rr_2x^{1/m} = \frac{3m(m-1)}{m+1}x^{(m-1)/m}, \qquad r_2x^{1/m} = \frac{2m(2m-1)}{m+1}x^{(m-1)/m},$$

$$a_4 = 4rr_3x^{1/m} = \frac{2(m+1)}{(m-1)(2m-1)}x^{1/m}, \qquad r_4x^{1/m} = \frac{(m+1)(2m+1)}{(m-1)(2m-1)}x^{1/m},$$

$$a_5 = 5rr_4x^{1/m} = \frac{5m(m-1)(2m-1)}{(m+1)(2m+1)}x^{(m-1)/m},$$

$$r_5x^{1/m} = \frac{3m(2m-1)(3m-1)}{(m+1)(2m+1)}x^{(m-1)/m}, \text{ etc.}$$

The expressions for the r's suggest the general relations.

$$r_{2n-1}x^{1/m} = \frac{nm(2m-1)(3m-1)\cdots(nm-1)}{(m+1)(2m+1)(3m+1)\cdots\{(n-1)m+1\}}\,x^{(m-1)/m},$$

and

$$r_{2n}x^{1/m} = \frac{(m+1)(2m+1)(3m+1)\cdots(nm+1)}{(m-1)(2m-1)(3m-1)\cdots(nm-1)}\,x^{1/m}.$$

These can be proved by induction.

We can now write the continued fraction for $f(x+h)=N^{1/m}=(\alpha^m+\beta)^{1/m}$ in the form

$$(2)\quad \alpha+\cfrac{\beta}{m\alpha^{m-1}+}\ \cfrac{\beta}{\cfrac{2}{m-1}\alpha+}\ \cfrac{\beta}{\cfrac{3m(m-1)}{m+1}\alpha^{m-1}+}\ \cfrac{\beta}{\cfrac{2(m+1)}{(m-1)(2m-1)}\alpha+}\cdots.$$

To transform this into a more convenient expression we note that if b_r, a_r and b_{r+1} in the continued fraction

$$a_0+\cfrac{b_1}{a_1+}\ \cfrac{b_2}{a_2+}\ \cfrac{b_3}{a_3+}\cdots\cfrac{b_r}{a_r+}\cdots$$

are all multiplied by the same number k, the value of the continued fraction is unchanged [2]. Also if $b_{r+1}k$, a_{r+1} and b_{r+2} in the new fraction are all divided by k, the value of the continued fraction is still unchanged. By a repetition of this process using different k's we obtain

$$a_0+\cfrac{b_1k_1}{a_1k_1+}\ \cfrac{b_2k_2}{\cfrac{k_2}{k_1}a_2+}\ \cfrac{b_3k_3}{\cfrac{k_1k_3}{k_2}a_3+}\ \cfrac{b_4k_4}{\cfrac{k_2k_4}{k_1k_3}a_4+}\cdots$$

as equivalent to the original fraction.

By applying this process to (2) with $k_1=1$, $k_2=m-1$, $k_3=m+1$, $k_4=2m-1$, $k_5=2m+1$, *etc.*, we obtain

$$(3)\ N^{1/m}=\alpha+\cfrac{\beta}{m\alpha^{m-1}+}\ \cfrac{\beta(m-1)}{2\alpha+}\ \cfrac{\beta(m+1)}{3m\alpha^{m-1}+}\ \cfrac{\beta(2m-1)}{2\alpha+}\ \cfrac{\beta(2m+1)}{5m\alpha^{m-1}+}\cdots,$$

wherein the numerators are β times the successive terms of the series

$$1,\quad m-1,\quad m+1,\quad 2m-1,\quad 2m+1,\quad 3m-1,\quad 3m+1,\quad \textit{etc.},$$

and the denominators are alternately $m\alpha^{m-1}$ times the successive terms of the series 1, 3, 5, 7, $\cdots$ and 2α.

If m is not integral but rational and of the form m/l we write the required $N^{l/m}$ in the form $(\alpha^{m/l}+\beta)^{l/m}$ choosing α so that it is an lth power, *e.g.*, if $36^{1/3}$

were required we could choose α to be 8 so that $36=8^{5/3}+4$, and then form the continued fraction for $N^{l/m}$ thus

$$N^{l/m} = \alpha + \frac{\beta l}{m\alpha^{(m-l)/l}+} \frac{\beta(m-l)}{2\alpha+} \frac{\beta(m+l)}{3m\alpha^{(m-l)/l}+} \cdots. \tag{4}$$

Some numerical examples are as follows.

$$2^{1/3} = 1 + \frac{1}{3+} \frac{2}{2+} \frac{4}{9+} \frac{5}{2+} \frac{7}{15+} \frac{8}{2+} \frac{10}{21+} \frac{11}{2+} \cdots,$$

$$2^{2/3} = 1 + \frac{2}{3+} \frac{1}{2+} \frac{5}{9+} \frac{4}{2+} \frac{8}{15+} \frac{7}{2+} \frac{11}{21+} \frac{10}{2+} \cdots,$$

$$131^{5/7} = 32 + \frac{15}{28+} \frac{6}{64+} \frac{36}{84+} \frac{27}{64+} \frac{57}{140+} \cdots.$$

References

1. L. M. Milne-Thomson, Calculus of Finite Differences, London, Macmillan 1933.
2. S. Barnard and J. M. Child, Advanced Algebra, London, Macmillan, 1937, p. 197.

ON A PROPERTY OF COMPLEX POWER SERIES

K. F. Moppert, University of Tasmania

Let $f(z)$ be a complex-valued function of the complex variable $z=x+iy$, defined for $|z|\leqq 1$. We define the set H as the set of functions f such that the Lebesgue-integral

$$\|f\|^2 = \iint_K |f(z)|^2 dxdy$$

exists, the integral being extended over the area of the real unit circle K, $x^2+y^2\leqq 1$ with functions f and g defining the same element of H if $f=g$ except on a set of plane measure zero. For two elements $f\in H$ and $g\in H$ we define the scalar-product

$$(f, g) = \iint_K f(z)\overline{g(z)}dxdy.$$

It is obvious that H is a Hilbert-space. If f is regular for $|z|\leqq 1$ then $f\in H$.

Let ν, μ be non-negative integers. Then we have

$$(z^\nu, z^\mu) = \iint_K z^\nu \bar{z}^\mu dxdy = \int_0^{2\pi} \int_0^1 r^{\nu+\mu+1} e^{i(\nu-\mu)\phi} dr d\phi = \begin{cases} 0 \ (\nu \neq \mu) \\ \dfrac{\pi}{\nu+1} (\nu = \mu). \end{cases}$$

Therefore, the set of functions

$$f_\nu(z) = \sqrt{\frac{\nu+1}{\pi}}\, z^\nu \qquad (\nu = 0, 1, \cdots)$$

is an orthonormal set in the sense that $(f_\nu, f_\mu) = \delta_{\nu\mu}$.

As an immediate consequence we see that if $f \in H$ and $f = \sum_0^\infty a_\nu z^\nu$, in the sense that $\|f - \sum_{\nu=0}^n a_\nu z^\nu\| \to 0$, then

$$a_\nu = \frac{\nu+1}{\pi}(f(z), z^\nu) = \frac{\nu+1}{\pi}\iint_K f(z)\bar{z}^\nu dxdy.$$

If $f = \sum_0^\infty a_\nu z^\nu$ is regular for $|z| \leqq 1$, this formula reduces to Cauchy's formula. In order to see this we transform the last integral as follows

$$\iint_K f(z)\bar{z}^\nu dxdy = \int_0^1\int_0^{2\pi} f(z) r^{\nu+1} e^{-i\nu\phi} d\phi dr = \int_0^1 r^{2\nu+1}\int_0^{2\pi} f(z) r^{-\nu} e^{-i\nu\phi} d\phi dr.$$

Here we have

$$\int_0^{2\pi} f(z) r^{-\nu} e^{-i\nu\phi} d\phi = \int_0^{2\pi} f(z) z^{-\nu} d\phi = -i\int f(z) z^{-(\nu+1)} dz,$$

the integral being taken along the circle $|z| = r$. Now, by Cauchy's theorem, this integral is independent of r, $r \leq 1$. Accordingly, the path of integration may be taken to be any closed rectifiable curve C contained in $|z| \leqq 1$. So we get

$$a_\nu = -\frac{i(\nu+1)}{\pi}\int_C f(z) z^{-(\nu+1)} dz \int_0^1 r^{2\nu+1} dr = \frac{1}{2\pi i}\int_C f(z) z^{-(\nu+1)} dz.$$

ANOTHER NOTE ON HERMITE POLYNOMIALS

M. P. Drazin, Trinity College, Cambridge

In a recent note (this Monthly, vol. 62, 1955, 646–647) L. Carlitz gave a new proof of the fact that the Hermite polynomials, defined as

$$H_m(x) = \sum_s (-1)^s (2x)^{m-2s} \frac{m!}{s!(m-2s)!},$$

satisfy the identities

$$H_m(x) H_n(x) = \sum_r 2^r r! \binom{m}{r}\binom{n}{r} H_{m+n-2r}(x) \qquad (m, n = 0, 1, \cdots).$$

We give here a very brief and elementary verification of these identities.

Writing out both sides (for given m, n) as explicit polynomials in x and comparing the coefficients of x^{m+n-2v} on either side (for a given non-negative integer $v \leq \frac{1}{2}(m+n)$), we see at once that the identities are equivalent to

$$\sum_{s+t=v} (-1)^{s+t} 2^{m+n-2s-2t} \frac{m!n!}{s!t!(m-2s)!(n-2t)!}$$

$$= \sum_{r+u=v} 2^r r! \binom{m}{r} \binom{n}{r} (-1)^u 2^{m+n-2r-2u} \frac{(m+n-2r)!}{u!(m+n-2r-2u)!},$$

i.e.

$$\sum_{s} \frac{(m+n-2v)!}{s!(v-s)!(m-2s)!(n-2v+2s)!} = \sum_{r} \frac{(-2)^r (m+n-2r)!}{r!(v-r)!(m-r)!(n-r)!},$$

which we may in turn rewrite in the binomial form

$$\sum_{s} \binom{v}{s} \binom{m+n-2v}{m-2s} = \sum_{r} (-2)^r \binom{v}{r} \binom{m+n-2r}{m-r} \tag{1}$$

$$(m, n = 0, 1, \cdots; v = 0, 1, \cdots, [(m+n)/2]).$$

But (even for arbitrary complex n, v) this is just what we get on comparing coefficients of x^m in the obvious relation

$$(1+x^2)^v (1+x)^{m+n-2v} = \left\{1 - \frac{2x}{(1+x)^2}\right\}^v (1+x)^{m+n}. \tag{2}$$

The special cases of (1), (2) corresponding to the substitution $n = 2v - m$ were noted by E. Netto in his *Lehrbuch der Combinatorik* (Leipzig, 1927); in particular, (1) becomes

$$\sum_{r} (-2)^r \binom{v}{r} \binom{2v-2r}{m-r} = \binom{v}{m/2},$$

where the sum is of course to be taken over all integral values of r for which

$$\binom{v}{r} \binom{2v-2r}{m-r}$$

is non-zero, and where the binomial coefficient on the right is to be interpreted as zero if m is odd. As T. S. Nanjundiah (this MONTHLY, vol. 61, 1954, 700–702) has shown, this in turn includes a formula of E. Grosswald (this MONTHLY, vol. 60, 1953, 179–181).

Finally, it is perhaps worth noting also the rather similar relation

$$(1+x^3)^v (1+x)^{m+n-3v} = \left\{1 - \frac{3x}{(1+x)^2}\right\}^v (1+x)^{m+n},$$

whence

$$\sum_s \binom{v}{s}\binom{m+n-3v}{m-3s} = \sum_r (-3)^r \binom{v}{r}\binom{m+n-2r}{m-r},$$

and in particular

$$\sum_r (-3)^r \binom{v}{r}\binom{3v-2r}{m-r} = \binom{v}{m/3}.$$

ON A GENERALIZATION OF THE GEOMETRIC SERIES

M. S. KLAMKIN, AVCO Research and Advanced Development, Lawrence, Mass.

In a previous paper, (this MONTHLY, vol. 56, p. 325), R. Stalley sums a generalized geometric series obtaining the following results:

$$(1)\quad K_n(x) = \sum_{s=1}^{\infty} s^n x^s = (1-x)^{-n-1} \sum_{r=1}^{n} \left[\sum_{m=1}^{r} (-1)^{m+1} \binom{n+1}{m-1} (r-m+1)^n \right] x^r$$

where $|x|<1$. In this note $K_n(x)$ will be obtained in a different form by an immediate application of a theorem of Montmort [1]. Also, it will be shown how to express the sum in terms of the generalized Bernoulli numbers or Stirling numbers of the first kind.

Montmort's theorem states that

$$(2)\qquad \sum_{r=1}^{\infty} a_r x^r = \left(\frac{x}{1-x}\right) a_1 + \left(\frac{x}{1-x}\right)^2 \Delta a_1 + \left(\frac{x}{1-x}\right)^3 \Delta^2 a_1 + \cdots,$$

where it is assumed that the series $\sum_{r=1}^{\infty} a_r x^r$ converges for $|x|<1$, and that also $|x|<|1-x|$. If $a_r = r^n$, then $\Delta^m a_1 = 0$ for $m \geqq n+1$. For this case, $\sum_{r=1}^{\infty} a_r x^r$ will be given in closed form, and there will be no need for the restriction $|x|<|1-x|$. Consequently,

$$(3)\qquad K_n(x) = \sum_{s=1}^{\infty} s^n x^s = (1-x)^{-n-1} \sum_{r=0}^{n} x^{r+1}(1-x)^{n-r} \Delta^r 1^n.$$

Another way of obtaining (3) would be to express s^n in the following form [2]:

$$(4)\qquad s^n = 1 + \binom{s-1}{1}\Delta 1^n + \binom{s-1}{2}\Delta^2 1^n + \cdots + \Delta^{s-1} 1^n.$$

Comparing (1) and (3), one obtains the identity

$$(5)\qquad \sum_{s=0}^{n} x^{s+1}(1-x)^{n-s} \Delta^s 1^n = \sum_{r=1}^{n} \sum_{m=1}^{r} (-1)^{m+1} \binom{n+1}{m-1} (r-m+1)^n x^r.$$

A direct proof of (5) is rather long but will lead to interesting results. It would

be quicker to show that each side of the equation satisfies the recurrence relation

$$K_{n+1}(x) = xK_n'(x), \tag{6}$$

which was the basis for obtaining (1). This is easily done using the properties of the difference operator Δ [4].

To prove (5) directly we use the relation

$$\Delta^s 1^n = \sum_{m=0}^{s} (-1)^m \binom{s}{m} (s-m+1)^n, \tag{7}$$

expand $(1-x)^{n-s}$ by the binomial theorem, and then after interchanging the order of summation and equating coefficients of like powers of x, we obtain the identity

$$(-1)^t \sum_{s=0}^{t-1} \sum_{m=0}^{s} (-1)^{s+m+1} \binom{n-s}{t-s-1} \binom{s}{m} (s-m+1)^n = \sum_{m=1}^{t} (-1)^{m+1} \binom{n+1}{m-1} (t-m+1)^n. \tag{8}$$

By letting $s-m+1=a$, and $t-m+1=b$, and interchanging order of summations, (8) is transformed into

$$\sum_{b=1}^{t} (-1)^b \binom{n+1}{t-b} b^n = \sum_{a=1}^{t} (-1)^a a^n \sum_{s=a}^{t} \binom{n-s+1}{t-s} \binom{s-1}{a-1}. \tag{9}$$

Consequently, it follows that

$$\begin{aligned} \binom{n+1}{t-b} &= \sum_{s=b}^{t} \binom{n-s+1}{t-s} \binom{s-1}{b-1} \\ &= \sum_{p=0}^{t-b} \binom{n-p-b+1}{t-p-b} \binom{p+b-1}{p}. \end{aligned} \tag{10}$$

Equation (10) is a known binomial identity [3]. We have thus shown that (10) follows from (1) and (3), and that also (3) and (10) imply (1).

To express $K_n(x)$ in terms of generalized Bernoulli numbers or Stirling numbers of the first kind we use the following relations [4], [5]:

$$\begin{aligned} \Delta^r 0^n + \Delta^{r+1} 0^n &= \Delta^r 1^n, \\ \Delta^r 0^n &= \frac{n!}{(n-r)!} B_{n-r}^{(-r)}, \\ S_n^m &= \binom{n-1}{m-1} B_{n-m}^{(n)}. \end{aligned} \tag{11}$$

References

1. G. Chrystal, Textbook of Algebra, Chelsea, 1952, vol. 2, p. 406.
2. H. S. Hall and S. R. Knight, Higher Algebra, Macmillan, 1932, p. 324.
3. W. Feller, An Introduction to Probability Theory and Applications, John Wiley, 1950, p. 48.
4. L. M. Milne-Thompson, The Calculus of Finite Differences, Macmillan, 1951, pp. 52, 134–135.
5. C. Jordan, Calculus of Finite Differences, Chelsea, 1947, p. 146.

ON THE SKEW QUADRANGLE*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. LEMMA. *Given two vectors $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$, and two points M and M' which divide $\overrightarrow{AB}$ and $\overrightarrow{B'A'}$ in the same ratio, let π' and π be planes through M and M' perpendicular to $A'B'$ and AB. If O' is any point in π', then a necessary and sufficient condition that a point O lies in π is that*

$$\overrightarrow{O'A}\cdot\overrightarrow{OA'} = \overrightarrow{O'B}\cdot\overrightarrow{OB'}.$$

Proof. If we set $MA/MB = M'B'/M'A' = k$, $(k \neq 1)$, the point M being the barycenter of points A and B with coefficients -1 and k, we obtain

$$\overrightarrow{O'M}\cdot(k-1) = \overrightarrow{O'B}\cdot k - \overrightarrow{O'A},$$

whence

$$\begin{aligned}\overrightarrow{O'M}\cdot\overrightarrow{A'B'}(k-1) &= (\overrightarrow{O'B}k - \overrightarrow{O'A})\cdot(\overrightarrow{OB'} - \overrightarrow{OA'}) \\ &= \overrightarrow{O'B}\cdot\overrightarrow{OB'}k - \overrightarrow{O'A}\cdot\overrightarrow{OB'} - \overrightarrow{O'B}\cdot\overrightarrow{OA'}k + \overrightarrow{O'A}\cdot\overrightarrow{OA'}.\end{aligned} \tag{1}$$

Symmetrically,

$$\overrightarrow{OM'}\cdot\overrightarrow{AB}(k-1) = \overrightarrow{OA'}\cdot\overrightarrow{O'B}k - \overrightarrow{OB'}\cdot\overrightarrow{O'B} - \overrightarrow{OA'}\cdot\overrightarrow{O'A}k + \overrightarrow{OB'}\cdot\overrightarrow{O'A} \tag{2}$$

On adding the relations (1), (2) term by term, one obtains, after division by $k-1$,

$$\overrightarrow{O'M}\cdot\overrightarrow{A'B'} + \overrightarrow{OM'}\cdot\overrightarrow{AB} = \overrightarrow{O'B}\cdot\overrightarrow{OB'} - \overrightarrow{O'A}\cdot\overrightarrow{OA'} \tag{3}$$

We recall that the scalar product of two vectors is zero if, and only if, either at least one of the vectors is null, or, the two vectors are perpendicular. If the point O' of π' is at M, the vector $\overrightarrow{O'M}$ is null; if it is distinct from M, the vectors $\overrightarrow{O'M}$ and $\overrightarrow{A'B'}$ are perpendicular, and $\overrightarrow{O'M}\cdot\overrightarrow{A'B'} = 0$. In order that the point O

* Translated by E. A. Nordhaus.

lie in the plane π, it is necessary and sufficient that $\overrightarrow{OM'}\cdot\overrightarrow{AB}=0$, so that by (3)

$$\overrightarrow{O'A}\cdot\overrightarrow{OA'}=\overrightarrow{O'B}\cdot\overrightarrow{OB'}.$$

COROLLARY. *Given two skew quadrangles $ABCD$, $A'B'C'D'$ with the points $M, N, P, Q, M', N', P', Q'$ taken on $AB, BC, CD, DA, A'B', B'C', C'D', D'A'$ so that*

$$\frac{MA}{MB}=\frac{M'B'}{M'A'}=k_1,\quad \frac{NB}{NC}=\frac{N'C'}{N'B'}=k_2,\quad \frac{PC}{PA}=\frac{P'D'}{P'D'}=k_3,\quad \frac{QA}{QD}=\frac{Q'D'}{Q'A'}=k_4,$$

if the planes through M, N, P, Q perpendicular to $A'B', B'C', C'D', D'A'$ are concurrent, then the planes through M', N', P', Q' perpendicular to AB, BC, CD, DA are also concurrent.

Proof. Let O' be the intersection of the planes through M, N, P, Q perpendicular to $A'B', B'C', C'D', D'A'$ and O the intersection of the planes through M', N', P' perpendicular to AB, BC, CD. From preceding remarks, we obtain

$$\overrightarrow{O'A}\cdot\overrightarrow{OA'}=\overrightarrow{O'B}\cdot\overrightarrow{OB'},\qquad \overrightarrow{O'B}\cdot\overrightarrow{OB'}=\overrightarrow{O'C}\cdot\overrightarrow{OC'},\qquad \overrightarrow{O'C}\cdot\overrightarrow{OC'}=\overrightarrow{O'D}\cdot\overrightarrow{OD'},$$

and, since $\overrightarrow{O'D}\cdot\overrightarrow{OD'}=\overrightarrow{O'A}\cdot\overrightarrow{OA'}$, the point O belongs to the plane through Q' perpendicular to DA.

2. A Particular Case (a) $k_1=k_2=k_3=k_4=k$.

LEMMA. *In order that the planes through the points M, N, P, Q (which divide the sides AB, BC, CD, DA in the ratio k) perpendicular to the corresponding sides of another skew quadrangle $A'B'C'D'$ shall be concurrent, it is necessary and sufficient that*

$$\overline{A'A}^2+\overline{B'B}^2+\overline{C'C}^2+\overline{D'D}^2)(1+k)$$

$$(4)\qquad =\overline{A'B}^2k+\overline{B'A}^2+\overline{B'C}^2k+\overline{C'B}^2+\overline{C'D}^2k+\overline{D'C}^2+\overline{D'A}^2k+\overline{A'D}^2.$$

Proof. Supposing that the planes through M, N, P, Q perpendicular to $A'B', B'C', C'D', D'A'$ are concurrent at point O', one obtains

$$\overline{MA'}^2-\overline{MB'}^2=\overline{O'A'}^2-\overline{O'B'}^2,\qquad \overline{NB'}^2-\overline{NC'}^2=\overline{O'B'}^2-\overline{O'C'}^2$$
$$\overline{PC'}^2-\overline{PD'}^2=\overline{O'C'}^2-\overline{O'D'}^2,\qquad \overline{QD'}^2-\overline{QA'}^2=\overline{O'D'}^2-\overline{O'A'}^2,$$

which yields, on adding these relations term by term,

$$(5)\qquad \overline{MA'}^2-\overline{MB'}^2+\overline{NB'}^2-\overline{NC'}^2+\overline{PC'}^2-\overline{PD'}^2+\overline{QD'}^2-\overline{QA'}^2=0.$$

Since $MA/MB=k$, M is the barycenter of A and B with coefficients 1 and

$-k$, and consequently

$$\overline{A'A^2} - \overline{A'B^2}k = \overline{A'M^2}(1-k) + \overline{MA^2} - \overline{MB^2}k,$$
$$\overline{B'A^2} - \overline{B'B^2}k = \overline{B'M^2}(1-k) + \overline{MA^2} - \overline{MB^2}k,$$

whence

$$\overline{MA'^2} - \overline{MB'^2} = \frac{1}{1-k}(\overline{A'A^2} - \overline{A'B^2}k - \overline{B'A^2} + \overline{B'B^2}k), \tag{6}$$

with analogous relations holding for $\overline{NB'^2}-\overline{NC'^2}$, $\overline{PC'^2}-\overline{PD'^2}$, $\overline{QD'^2}-\overline{QA'^2}$, which, when added to (6) give, using (5), the relation (4), and conversely.

COROLLARY. *If the planes through points M, N, P, Q (which divide the edges AB, BC, CD, DA of a skew quadrangle $ABCD$ in the ratio k) perpendicular to the corresponding edges of another quadrangle $A'B'C'D'$ are concurrent, then the planes through the points M', N', P', Q' (which divide the edges $B'A'$, $C'B'$, $D'C'$, $A'D'$ in the same ratio k) perpendicular to the corresponding sides of the quadrangle $ABCD$ are also concurrent* [1].

Proof. By the preceding lemma, in order that the planes through M', N', P', Q' and perpendicular to AB, BC, CD, DA be concurrent, it is necessary and sufficient that

$$\overline{(AA'^2} + \overline{BB'^2} + \overline{CC'^2} + \overline{DD'^2})(k+1) = \overline{AB'^2} + \overline{BA'^2}k + \overline{BC'^2} + \overline{CB'^2}k + \overline{CD'^2} + \overline{DC'^2}\cdot k + \overline{DA'^2} + \overline{AD'^2}\cdot k.$$

This is the relation (4).

(b) Provided $k=-1$, the points M, N, P, Q, M', N', P', Q' are the midpoints of the corresponding edges of the quadrangles $ABCD$, $A'B'C'D'$, which are called *orthologic quadrangles relative to the midpoints of corresponding edges* [2], we may define them in the following manner:

In order that two skew quadrangles $ABCD$ and $A'B'C'D'$ shall be orthologic relative to the midpoints of corresponding edges, it is necessary and sufficient that

$$\overline{AB'^2} - \overline{BA'^2} + \overline{BC'^2} - \overline{CB'^2} + \overline{CD'^2} - \overline{DC'^2} + \overline{DA'^2} - \overline{AD'^2} = 0.$$

COROLLARY. *In order that two skew quadrangles $ABCD$ and $A'B'C'D'$ shall be orthologic relative to the midpoints of corresponding edges, it is necessary and sufficient that*

$$\overline{B'D'}\cdot\overline{A_1C_1} = \overline{A'C'}\cdot\overline{B_1D_1} \quad \text{or} \quad \overline{BD}\cdot\overline{A_1'C_1'} = \overline{AC}\cdot\overline{B_1'D_1'},$$

where A_1 and C_1, B_1 and D_1, A_1' and C_1', B_1' and D_1' designate the orthogonal projections of A and C, B and D, A' and C', B' and D' on $B'D'$, $A'C'$, BD, AC.

Proof. If m' denotes the midpoint of $B'D'$, we obtain first

$$\overline{AB'^2} - \overline{AD'^2} = 2\overline{B'D'}\cdot\overline{m'A_1}, \qquad \overline{CD'^2} - \overline{CB'^2} = 2\overline{D'B'}\cdot\overline{m'C_1},$$

and then, by analogy,

$$\overline{AB'^2} - \overline{AD'^2} + \overline{CD'^2} - \overline{CB'^2} = 2\overline{B'D'} \cdot \overline{C_1A_1},$$
$$\overline{BC'^2} - \overline{BA'^2} + \overline{DA'^2} + \overline{DC'^2} = 2\overline{A'C'} \cdot \overline{B_1D_1},$$

so that

$$\overline{B'D'} \cdot \overline{A_1C_1} = \overline{A'C'} \cdot \overline{B_1D_1}.$$

3. In the discussion which has preceded, the corresponding edges of the quadrangles $A'B'C'D'$ and $ABCD$ may be replaced by the opposite edges.* In the particular case in which $k_1=k_2=k_3=k_4=k=-1$, the skew quadrangles $ABCD$ and $A'B'C'D'$ are, at the same time, orthologic relative to the midpoints of corresponding edges and relative to the midpoints of opposite edges.

References

1. R. Blanchard, Mathesis, vol. 56, p. 352, question 3355.

2. V. Thébault, Parmi les belles figures de la géométrie dans l'espace, p. 3, Librairie Vuibert, Paris, 1955.

AN IDENTITY FOR THE SUM OF MULTINOMIAL COEFFICIENTS

R. C. Kao, RAND Corporation, Santa Monica and L. H. Zetterberg, FOA 3, Stockholm, Sweden

1. Summary. Let r, n be positive integers with $r \geqq n$, and let $\{k_i\}$, $i=1, \cdots, n$, be a proper n-fold partition of r. A simple formula (2.6) is given for the sum of multinomial coefficients of degree r over all proper n-fold partitions of r. A second proof is obtained by the explicit construction of two probability models (Theorems 3.1 and 3.2) for the same "collector's problems."†

2. Sum of Multinomial Coefficients. If r and n are positive integers ($r \geqq n$), an *n-fold partition* $\sigma^*_{n,r}$ of r is an ordered set of nonnegative integers k_i ($i=1, \cdots, n$) such that $\sum_{i=1}^{n} k_i = r$. A *proper n-fold partition* $\sigma_{n,r}$ of r is a $\sigma^*_{n,r}$ with the further restriction: $k_i \geqq 1$ ($i=1, \cdots, n$). For given r and n, let $\mathfrak{S}^*_{n,r}$ be the set of all n-fold partitions of r, and $\mathfrak{S}_{n,r}$ be the set of all proper n-fold partitions of r.

By a *multinomial coefficient of degree r* is meant any coefficient in the expansion of the following power series in the indeterminates $x_1, \cdots, x_n$:

$$(2.1) \qquad (x_1 + \cdots + x_n)^r = \sum_{\sigma^*_{n,r} \in \mathfrak{S}^*_{r,n}} \frac{r!}{\prod_{i=1}^{n} k_i!} x_1^{k_1} \cdots x_n^{k_n}.$$

* Thus, the edges $C'D'$, $D'A'$, $A'B'$, $B'C'$ of the quadrangle $A'B'C'D'$ are associated with the edges AB, BC, CD, DA and *vice versa*.

† See W. Feller, An Introduction to Probability Theory and Its Applications, Vol. I, John Wiley, 1950, pp. 63–64.

The problem under consideration is to find the sum

$$\sum_{\sigma_{n,r}\in\mathfrak{S}_{n,r}} \frac{r!}{\prod_{i=1}^{n} k_i!}. \tag{2.2}$$

LEMMA 2.1. *For* $0\leqq q\leqq n-1$,

$$S_{n,q} = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^q = \begin{cases} (-1)^{n+1} & \text{if } q = 0, \\ 0 & \text{if } 1 \leqq q \leqq n-1. \end{cases} \tag{2.3}$$

*Proof.** From the identity

$$(n-i)^q = n(n-i)^{q-1} - i(n-i)^{q-1}, \tag{2.4}$$

it follows that the quantity $S_{n,q}$ defined by (2.3) satisfies the equation

$$S_{n,q} = nS_{n,q-1} + nS_{n-1,q-1}. \tag{2.5}$$

Since $S_{n,0}=(-1)^{n+1}$ by definition, we have immediately $S_{n,q}=0$ for $1\leqq q\leqq n-1$.

THEOREM 2.2. *For* $r\geqq n$,

$$\sum_{\sigma_{n,r}\in\mathfrak{S}_{n,r}} \frac{r!}{\prod_{i=1}^{n} k_i!} = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^r. \tag{2.6}$$

Proof. We first proceed by induction on n. For $n=1$ and arbitrary r, (2.6) becomes

$$r!/r! = 1 = 1^r = \sum_{i=0}^{0} (-1)^i \binom{1}{i} (1-i)^r. \tag{2.7}$$

Assume the theorem true for all r', n' such that $0\leqq r'\leqq r$, $0\leqq n'\leqq n$ with $r'\geqq n'$. Consider the left side of (2.6) for r and $n+1$, where $r\geqq n+1$. Then, since $\sum_{i=1}^{n} k_i\geqq n$ by hypothesis, it follows that $1\leqq k_{n+1}\leqq r-n\leqq r-\sum_{i=1}^{n} k_i$ and $r'=r-k_{n+1}\leqq r$. We have

$$\begin{aligned} \sum_{\sigma_{n+1,r}\in\mathfrak{S}_{n+1,r}} \frac{r!}{\prod_{i=1}^{n+1} k_i!} &= \sum_{k_{n+1}=1}^{r-n} \sum_{\sigma_{n,r-k_{n+1}}\in\mathfrak{S}_{n,r-k_{n+1}}} \frac{r!}{\prod_{i=1}^{n} k_i! k_{n+1}!} \\ &= \sum_{k_{n+1}=1}^{r-n} \binom{r}{k_{n+1}} \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^{r-k_{n+1}}. \end{aligned} \tag{2.8}$$

By Lemma 2.1, the inner sum is zero for $k_{n+1}=r-n+1, \cdots, r-1$ so that we may extend the upper limit of summation of k_{n+1} to $r-1$ without changing the

* We are indebted to the referee for this simpler proof. Our original proof was longer and based on differentiation of power series. Also, *ibid.*, p. 77.

value of the entire sum. Adding and subtracting terms corresponding to $k_{n+1}=0, r$ and interchanging the order of summation, we get for (2.8)

$$
\begin{aligned}
&\sum_{\sigma_{n+1,r}\in\mathfrak{S}_{n+1,r}} \frac{r!}{\prod_{i=1}^{n+1} k_i!} \\
&\quad = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \sum_{k_{n+1}=0}^{r} \binom{r}{k_{n+1}} (n-i)^{r-k_{n+1}} \\
&\quad\quad - \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^r + (-1)^n \\
&\quad = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n+1-i)^r - \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^r + (-1)^n.
\end{aligned} \tag{2.9}
$$

Changing $i=j-1$ in the second term and using the identity

$$
\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i},
$$

we get

$$
\sum_{\sigma_{n+1,r}\in\mathfrak{S}_{n+1,r}} \frac{r!}{\prod_{i=1}^{n+1} k_i!} = \sum_{i=0}^{n} (-1)^i \binom{n+1}{i} (n+1-i)^r. \tag{2.10}
$$

For induction on r, we have to show that (2.6) holds for $r=1$; and if it holds for all r', n' such that $0 \leqq r' \leqq r$, $0 \leqq n' \leqq n$ with $r' \geqq n'$, then it also holds for $r+1$ and n. The case $r=1$ is trivial. For the rest of the proof, we proceed thus: for $n=1$, the theorem holds for *arbitrary* r as the computation (2.7) shows. For this fixed n, we replace r by $r+1$ and then repeat the above induction on n; so the theorem is proved.

3. Probability Models. We now offer a second proof for Theorem 2.2 by constructing two different models for the following probability problem: Random sampling with replacement is made from a population of n objects, we seek the distribution of the random variable $X=$the number of drawings it takes to get objects of all kinds. Let $p_r = \Pr\{X=r\}$. Clearly, $p_r \equiv 0$ for $r<n$.

THEOREM 3.1.

$$
p_r = \frac{1}{n^r} \cdot \sum_{i=1}^{n} (-1)^{i-1} i \binom{n}{i} (n-i)^{r-1}. \tag{3.1}
$$

Proof. Define the sample space $\Omega_{n,r}$ to be the set of all (ordered) r-tuples, each coordinate of which may take on the values $1, \cdots, n$ (a set of labels for the n objects). $\Omega_{n,r}$ has therefore n^r points. Define a sequence of events A_i

$(i=1, \cdots, n)$ with A_i=the event that object i is not drawn in $r-1$ trials. Then,

$$p_r = \frac{1}{n} \cdot \Pr\{\text{exactly one of the } A_i\text{'s occurs}\} = \frac{1}{n} \cdot P_{[1]}. \tag{3.2}$$

By a well-known formula,*

$$P_{[1]} = S_1 - 2S_2 + 3S_3 - \cdots \pm nS_n \tag{3.3}$$

where

$$S_1 = \sum_i p_i = \sum_i \Pr\{A_i\}, \qquad S_2 = \sum_{i<j} p_{i,j} = \sum_{i<j} \Pr\{A_iA_j\}, \cdots. \tag{3.4}$$

In our problem,

$$S_i = \binom{n}{i}\binom{n-i}{n}^{r-1} \qquad (i = 1, \cdots, n),$$

so that (3.1) immediately follows from (3.2) after computing $P_{[1]}$.

THEOREM 3.2.

$$p_r = \frac{1}{n^{r-1}} \cdot \sum_{\sigma_{n-1,r-1} \in \mathfrak{S}_{n-1,r-1}} \frac{(r-1)!}{\prod_{i=1}^{n-1} k_i!}. \tag{3.5}$$

Proof. Let $\Omega_{n,r}$ be defined as in Theorem 3.1, each point of which has probability $1/n^r$ by hypothesis. Consider the set of all points or sequences in $\Omega_{n,r}$ with one of the labels $1, \cdots, n$ appearing for the first time in the sequence at the last position. For this *fixed* label, calculate the number of sequences with $n-1$ remaining labels in $r-1$ positions. This number is exactly

$$A = \sum_{\sigma_{n-1,r-1} \in \mathfrak{S}_{n-1,r-1}} \frac{(r-1)!}{\prod_{i=1}^{n-1} k_i!}. \tag{3.6}$$

Since any one of the labels can be placed at the last positions, we find

$$p_r = \frac{A}{n^r} \cdot n = \frac{1}{n^{r-1}} \sum_{\sigma_{n-1,r-1} \in \mathfrak{S}_{n-1,r-1}} \frac{(r-1)!}{\prod_{i=1}^{n-1} k_i!}. \tag{3.7}$$

Equating (3.1) and (3.5), we see that Theorem 2.2 follows after simple substitutions.

* Cf. *ibid.*, p. 64.

4. Further Extensions. The boundary condition: $k_i \geqq 1 (i = 1, \cdots, n)$ on $\mathfrak{S}_{n,r}$ may be generalized to other cases:

(a) $k_i \geqq a$ for $1 \leqq a \leqq r$; (b) $k_i \geqq a_i$ for $1 \leqq a_i \leqq r$;

(c) $b_i \geqq k_i \geqq a_i$ for $1 \leqq a_i \leqq b_i \leqq r$; (d) $\sum_{j=1}^{i} k_j \geqq a_i$ for $1 \leqq a_i \leqq r$; $(i = 1, \cdots, n)$.

The formulas for the sum of multinomial coefficients corresponding to these more general boundary conditions on k_i remain to be worked out. In each case, a meaningful collector's problem can be defined.

ON SYLVESTER'S LAW OF NULLITY

KURT BING, Rensselaer Polytechnic Institute

When one extends Sylvester's Law of Nullity* from square matrices over a field of rectangular ones,† the question arises which are the most general conditions. The answer is as follows.

Sylvester's Law of Nullity. *The nullity of AB is less than or equal to the sum of the nullities of A and B. It is greater than or equal to the nullity of A; it is greater than or equal to that of B provided A does not have more columns than it has rows.*

Proof. Let $R(M)$ and $N(M)$ denote the rank and the nullity of the matrix M, and let A and B be $m \times n$ and $n \times p$ matrices respectively.

(a) Following standard procedure,*,† choose non-singular $m \times m$ and $n \times n$ matrices P and Q such that the $m \times n$ matrix $PAQ = A^*$ is diagonal, having the entries of an identity matrix of rank $r = R(A)$ in the left upper corner and zeros elsewhere. Set $B^* = Q^{-1}B$, then A^*, B^*, $A^*B^* = PAB$ are equivalent to, and therefore have the ranks and nullities of A, B, and AB, respectively. The first r rows of A^*B^* are those of B^*, and the remaining $m - r$ rows consist of zeros; hence A^*B^*, which lacks $n - r$ possibly independent rows of B^*, has a rank of at least $R(B^*) - (n - R(A^*))$. Therefore $N(A^*B^*) = m - R(A^*B^*) \leqq m - R(A^*) + n - R(B^*) = N(A^*) + N(B^*)$.

(b) The null space of A is clearly a subspace of that of AB, hence $N(A) \leqq N(AB)$.

(c) Since $R(AB) \leqq R(B)$, we have $N(B) \leqq m - R(B) \leqq N(AB)$ provided $n \leqq m$. We may have $N(AB) < N(B)$ if $n > m$, as is shown by the example

$$A = [1\ 0], \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad AB = A, \qquad N(AB) = 0, \qquad N(B) = 1.$$

* See, for instance, M. Bôcher, Introduction to higher algebra, New York, 1907, pp. 78 and 80.

† See G. Birkhoff and S. Mac Lane, A survey of modern algebra, revised edition, New York, 1953, p. 235.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

ON THE DIOPHANTINE EQUATION $x^2+y^2+z^2+2xyz=1$

A. OPPENHEIM, University of Malaya, Singapore

In his note "A Diophantine equation characterizing the law of cosines" (this MONTHLY, vol. 62, 1955, pp. 251–252) Barnett gives the most general rational solution of the cubic equation

$$x^2 + y^2 + z^2 + 2xyz = 1, \tag{1}$$

in the form

$$x = \frac{b^2 + c^2 - a^2}{2bc}, \quad y = \frac{c^2 + a^2 - b^2}{2ca}, \quad z = \frac{a^2 + b^2 - c^2}{2ab}, \tag{2}$$

where a, b, c are any integers.

Now (1) has an infinity of rational integral solutions, *e.g.*

$$x = 16u^5 - 20u^3 + 5u, \quad y = 3u - 4u^3, \quad z = u^2 - 1, \tag{3}$$

where u is any integer. However, it is not obvious what values of a, b, c produce this solution. It is of some interest then to give the general solution of (1) in rational integers.

THEOREM. *Apart from the trivial solutions* $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$, *all integral solutions of the Diophantine equation*

$$x^2 + y^2 + z^2 + 2xyz = 1,$$

are given by the following rule: let p, q, r be any integers with greatest common divisor unity such that one of them is equal to the sum of the other two; let $u \geqq 1$ be any integer: then, writing ch for cosh,

$$x = \pm\, ch(p\theta), \quad y = \pm\, ch(q\theta), \quad z = \pm\, ch(r\theta), \tag{4}$$

where $\theta = \log\,(u+(u^2-1)^{1/2})$, are integers which satisfy the Diophantine equation provided the ambiguous signs are chosen to have product -1.

That x, y, z given by (4) do satisfy (1) is clear since

$$ch^2\,\alpha + ch^2\,\beta + ch^2\,\gamma - 2\,ch\,\alpha\,ch\,\beta\,ch\,\gamma = 1,$$

for any numbers α, β, γ such that $\alpha+\beta+\gamma=0$.

Again, for any integer $n \geqq 1$, $ch(n\theta)$ is an integral polynomial of degree n in $ch\,\theta$. In the present case $ch\,\theta = u$, an integer, so that x, y, z given by (4) are rational integers which satisfy (1). (It may be noted that the trivial solutions are also included in these polynomials for the value $u=0$.)

It remains to show that every non-trivial solution in integers of (1) can be put into the form (4). For this purpose note that (1) is an integral quadratic in each of x, y, z with leading coefficients which divide the coefficients of xyz. Hence, from any integral triad (x, y, z) satisfying (1), can be derived associated integral solutions (x', y, z), (x, y', z), (x, y, z') by the rule

$$x + x' = -2yz, \qquad (xx' = y^2 + z^2 - 1) \text{ etc.}$$

Thus any admissible triad gives rise to a class or chain (finite or infinite) of admissible triads: any two members of the chain are derivable one from the other by repeated use of association.

Now if $x = \pm ch\,\theta$, $y = \pm ch\,\phi$, $z = \pm ch\,\psi$ where $\theta+\phi+\psi=0$, then clearly

$$x' = \pm 2ch\,\phi\,ch\,\psi \mp ch\,\theta = \pm ch(\psi - \phi)$$

(the product of the ambiguous signs being -1, it follows that $xx'>0$) so that the associate solution (x', y, z) has the same form as (x, y, z). Thus if one member of a chain has the form (4), every member has this form.

Next suppose that there is an integral solution (x, y, z) such that (without loss of generality) $|x| \geqq |y| \geqq |z| \geqq 1$.

If $|z| > 1$, *then the associate solution* (x', y, z) *is such that*

$$|x| > |y| > |x'|.$$

For consider the quadratic in t, $f(t) \equiv t^2+y^2+z^2+2tyz-1=0$ for which $f(x)=0$, $f(x')=0$, $f(0)=y^2+z^2-1>0$. Plainly

$$\begin{aligned} f(-y \operatorname{sgn} z) &= 2y^2 + |z|^2 - 2y^2|z| - 1 \\ &= -(|z| - 1)(2y^2 - 1 - |z|) < 0, \end{aligned}$$

since $|y| \geqq |z| > 1$. Hence $|x| > |y| > |x'|$.

This descent can be carried on so long as min $(|x|, |y|, |z|) > 1$, which can hold for only a finite number of steps. Thus a solution must be reached in the chain for which $|z| = 1$ (or $|x| = 1$ or $|y| = 1$). If $z=1$, then $x=u$, $y=-u$: if $z=-1$, then $x=u$, $y=u$, where u is any integer not zero. But each of these solutions has the form of the theorem with $p=1$, $q=-1$, $r=0$.

It follows that all integral solutions of (1) apart from the trivial ones are given by (4).

It may be noted that $(\pm 1, 0, 0)$ (and its permutations), $(-1, -1, -1)$, $(-1, 1, 1)$, *etc.* give rise only to finite chains. Any integer u such that $|u| \geqq 2$ gives rise to the basic solutions $(u, -u, 1)$, $(u, u, -1)$ (and permutations), each of which generates an infinite chain.

The method of descent used here was first employed by Markoff [3] for the Diophantine equation $x^2+y^2+z^2=3xyz$ for which the only basic solutions are $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, and $(-1, -1, 1)$. Hurwitz [2] later applied it to the case of n variables,

$$x_1^2 + x_2^2 + \cdots + x_n^2 = tx_1x_2 \cdots x_n,$$

the main result being roughly that if $|t|$ is too large any basic solution must have one of $x_1, \cdots, x_n$ zero.

Reference may also be made to Mills [4] and to Barnes [1] who considered respectively the equations

$$x^2 + y^2 + ax + ay + 1 = txy,$$

$$x^2 + y^2 + c = txy.$$

It is plain that the method applies equally to an equation such as

$$2x_1^2 - 3x_2^2 + 4x_3^2 + 5x_4^2 + 3x_3x_4 + 2x_1 + 3x_2 + 5$$
$$= 60ax_2x_3x_4 + 12bx_1x_2x_3 + 30cx_1x_2x_4 + 20dx_1x_3x_4 + 60tx_1x_2x_3x_4$$

which has the property that it is a quadratic with integral coefficients in each of the variables such that if one of the roots in x_1, say, is integral, so also is the other. It can be shown that if $|t|$ is too large then any basic solution must have at least one of $x_1, \cdots, x_4$ zero.

Added in proof: I regret that I have overlooked the valuable paper of L. J. Mordell, *J. London Math. Soc.*, vol. 28, 1953, pp. 500–510.

References

1. E. S. Barnes, J. London Math. Soc., vol. 28, 1953, pp. 242–244.
2. A. Hurwitz, Mathematische Werke, vol. II, pp. 412–421.
3. A. Markoff, Math. Annalen, vol. 17, 1880, pp. 379–399.
4. W. H. Mills, Pacific J. Math., vol. 3, 1953, pp. 209–220.

A LOW ENERGY PROOF OF THE RECIPROCITY LAW

D. H. LEHMER, University of California

Most textbooks in the theory of numbers give a proof of the Law of Quadratic Reciprocity based on the celebrated Lemma of Gauss which he used in the so-called third proof of the theorem. An examination of these proofs shows at least in most cases, that in the deduction of the theorem from the lemma, certain by-products are obtained that are not really needed for the conclusion. These extra results, which are usually properties of the greatest integer function, are of some interest in themselves and stimulate the reader's interest in related problems. However, one may take the point of view that the reader is interested in reaching the conclusion without having to waste energy in passing over summits from which views may be had of the surrounding terrain. The proof of the Reciprocity Law given below is intended to be one of minimum energy in which the possibly unwilling reader is led along a path of constant gradient. The attempt is made to translate the reciprocal relationship stated in the theorem into the more elementary notions of geometrical symmetry. In

concocting this proof, the writer has borrowed ideas from a number of mathematicians. Besides Gauss, these include Eisenstein, Dedekind, Zeller, Frobenius, and Vandiver. For completeness the Lemma of Gauss is included in the presentation.

In order to introduce a measure of symmetry we make free use of the notion of the absolutely least remainder of A (mod m), that is the integer less than $m/2$ in absolute value which is congruent to A mod m, an odd integer. According to Euler's criterion, Legendre's symbol $(k|p)$ may even be defined as the absolutely least remainder of $k^{(p-1)/2}$ (mod p), an odd prime. It is then a theorem that, if k is not divisible by p, the symbol has the value $+1$ or -1 according as the quadratic congruence $x^2 \equiv k$ (mod p) has or has not solutions.

The Lemma of Gauss may be stated as follows:

Let p be an odd prime not dividing k and let μ be the number of numbers in the set

$$k, 2k, 3k, \cdots, \frac{p-1}{2}k, \tag{1}$$

whose absolutely least remainders (mod p) are negative. Then $(k|p) = (-1)^{\mu}$.

Proof. A typical element rk of (1) when reduced to its absolutely least remainder (mod p) can be written $rk \equiv \epsilon(r, k)r'$ (mod p) where r', like r, is a positive integer $< p/2$ and $\epsilon(r, k)$ is either $+1$ or -1. Taking the product of all such congruences and noting that the r' are distinct we find, after cancelling $\{(p-1)/2\}!$, that

$$k^{(p-1)/2} \equiv \prod_{r=1}^{(p-1)/2} \epsilon(r, k) \equiv (-1)^{\mu} \pmod{p}.$$

From the above remarks about Legendre's symbol, the lemma follows at once.

The Reciprocity Law may be stated as follows.

THEOREM. *Let p and q be odd, distinct primes. Then the symbols $(p|q)$ and $(q|p)$ are equal except when p and q are both of the form $4x-1$.*

Proof. To save repeating words, we shall speak of the case in which one or both of p and q are of the form $4x+1$ as Case I and the opposite situation in which both p and q are of the form $4x-1$ as Case II.

To compare the symbols $(p|q)$ and $(q|p)$ we may apply the Lemma of Gauss twice, once to the set

$$q, 2q, \cdots, \frac{p-1}{2}q \pmod{p}, \tag{2}$$

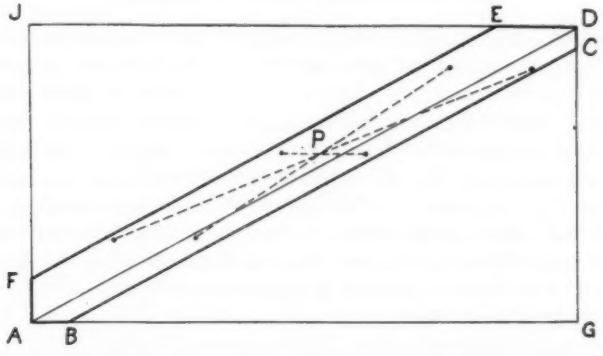
and again to the set

$$p, 2p, \cdots, \frac{q-1}{2}p \pmod{q}. \tag{3}$$

Denoting the number of negative absolutely least remainders obtained in each case by μ_1 and μ_2 respectively, and noting that $(p|q)(q|p)=(-1)^{\mu_1+\mu_2}$ is 1 or -1 according as the symbols are equal or not, we have only to prove the statement

(4) $\mu_1+\mu_2$ is even in Case I and odd in Case II.

In the first quadrant of the (x, y) plane we consider the hexagonal region H whose vertices $ABCDEF$ lie on the rectangle $AGDJ$ bounded by the coordinate axes and the lines $x=p/2$ and $y=q/2$, as shown in the figure. The long sides FE



and BC are formed by the lines $y=(q/p)x+1/2$, $x=(p/q)y+1/2$, respectively which are parallel to the diagonal AD and have intercepts $1/2$ on the axes. Points (x, y) interior to H are therefore characterized by the inequalities

$$(5)\qquad \begin{aligned} &0<x<p/2, \qquad && 0<y<q/2,\\ &y<qx/p+1/2, \qquad && y>qx/p-q/2p, \end{aligned}$$

and lie above or below the diagonal AD according as $y>qx/p$ or $y<qx/p$. We consider now the lattice points interior to H, that is the points which have integer coordinates (m, n). If the point (m, n) belongs to H so also does $((p+1)/2-m, (q+1)/2-n)$ as may be seen by substituting the coordinates of the latter point into the characteristic inequalities (5). The midpoint of these two points is the point P: $((p+1)/4, (q+1)/4)$, which is therefore the point about which the lattice points of H are symmetric. But P is a lattice point in Case II only. Hence we can make the statement

(6) The number of lattice points of H is even in Case I and odd in Case II.

Now the lattice points of H are divided into two classes by the diagonal AD. It is clear that there is no lattice point on the diagonal. Consider now a lattice point (m, n) below the diagonal so that $qm/p-q/2p<n<qm/p$ or $-q/2<pn-qm<0$. This exhibits an element np of (3) which has a negative absolutely least remainder (mod q). Conversely, such an element of (3) will graph into a

lattice point below the diagonal.

To consider the lattice points above the diagonal we have only to view our rectangle from its side AJ, interchanging axes and p and q, and to repeat the above reasoning. Thus we have proved that there are $\mu_1+\mu_2$ lattice points in H. Statement (4) now follows from (6) and the theorem is proved.

INTERVAL-ADDITIVE PROPOSITIONS

L. R. FORD, Illinois Institute of Technology

The present paper results from an attempt to unify the proofs of a large number of theorems connected with closed intervals. The basic properties of continuous functions in an interval, for example, or the facts about sets of points, are sometimes treated by appeals to the Heine-Borel theorem, sometimes by repeated divisions into subintervals, or by other devices. The concept to be presently introduced is believed to be generally rather easy to apply and to be comprehensible to a student at a fairly early level of mathematical attainment.

1. Interval-Additive propositions. A statement P concerning intervals will be called *interval-additive* if whenever P is true for each of two overlapping intervals (with common interior points) it is also true for the interval obtained by combining them; that is, their union.

As an example, the proposition that an interval is more than one unit long is interval-additive, whereas the proposition that it is less than one unit long, is not. The proposition that for a given number M a function satisfies $f(x)<M$ is interval-additive, but the proposition that $\int f(x)dx<M$, the integral being taken over the interval, is not. The proposition that a function is integrable, or continuous, or positive or of bounded variation, or strongly monotonic in an interval is interval-additive.

2. Propositions true at a point. We shall be concerned with a closed base interval $a\leqq x\leqq b$, which we shall call I. We shall say that a proposition P is true at a point x_0 of this interval, if it is true in some subinterval I_0 of I enclosing x_0. Here x_0 is an interior point of I_0 in general, but the subinterval may be closed or open. However, if x_0 is an end point a or b, which cannot be interior to a subinterval, we say that P is true there, if it is true in a subinterval terminating at, and including the end point.

We wish to proceed from truth at each point of an interval to truth in the whole interval. Our fundamental theorem is the following.

THEOREM. *If P is an interval-additive proposition which is true at each point of a closed interval $a\leqq x\leqq b$, then P is true in the whole interval.*

By hypothesis P is true in a subinterval ac, including the end point a. Consider all such subintervals in which P is true, and let u be the least upper bound of c. It is given that P is true at u.

If $u<b$, an interval about u combines with some ac_0 to produce an interval

ac extending to the right of u in which P is true. This is impossible, since u was defined as the least upper bound of c.

So $u=b$, and P is true in an interval ac which is either the closed interval I, or I with the end point b excluded, or a subinterval with c as near b as we like. In the latter cases a subinterval $a'b$ (including b) combines with ac to establish the truth of P in the whole closed interval.

The usefulness of this theorem arises from the fact that interval-additivity is a rather widespread phenomenon and that it is frequently easy to detect the additive character of a proposition. It is often so nearly obvious that it need merely be mentioned.

3. Functions continuous in a closed interval. Let $f(x)$ be continuous at all points of a closed interval I, $a \leqq x \leqq b$. We shall use the preceding theorem to derive certain of its properties.

About each x_0 and for any given $\epsilon_0 > 0$ there is a subinterval I_0 within which $|f(x)-f(x_0)| < \epsilon_0$, with the convention previously made for $x_0=a$ and $x_0=b$.

(a) We observe that $f(x)$ is *bounded* in each I_0, and boundedness is obviously an interval-additive property. Hence, $f(x)$ is bounded in I.

Here we have not used the fact that ϵ_0 can be made arbitrarily small. It suffices for boundedness in I that we have boundedness at each point; in other words, that $f(x)$ have a finite oscillation at each x_0.

(b) In I then the values of $f(x)$ have a least upper bound M and a greatest lower bound m. We show that these values are actually taken on by the function.

Suppose, on the contrary, that $f(x)$ is nowhere equal to M in I. Then ϵ_0 can be chosen so that in the resulting I_0 we have $f(x) < M_0 < M$ for some M_0; for example, $M_0 = \frac{1}{2}[M+f(x_0)]$. Now the statement that $f(x)$ is less than a number less than M in an interval is clearly interval-additive. Hence we have, for some M', $f(x) < M' < M$ in I, and the least upper bound is less than M. This contradiction establishes the result. That $f(x)=m$ somewhere in I is proved similarly.

An alternative proof consists in observing that if the theorem is not true the function $1/[M-f(x)]$ is continuous in I and hence bounded, $1/[M-f(x)] < K$. Then $f(x) < M - 1/K$, which is impossible.

(c) The function $f(x)$ takes on in I each value between M and m.

Suppose, on the contrary, that $f(x)$ is nowhere equal to an intermediate value n. Then $f(x)-n$ is nowhere zero. About each x_0 we can construct I_0 throughout which $f(x)-n$ is positive or is negative. Now, to say that a function retains the same sign throughout an interval is an interval-additive proposition. It follows that $f(x)-n$ has the same sign throughout I. But since $f(x)-n$ has the values $M-n$ and $m-n$, which differ in sign, this situation is impossible, and the theorem is true.

(d) A property of the function that is useful in proving the existence of the integral is the following: Given $\epsilon > 0$, it is possible to divide I into a finite number of abutting subintervals in each of which the oscillation of $f(x)$ is less than ϵ.

That it is possible so to divide an interval is an interval-additive statement.

For if each of two overlapping intervals can be so divided and if we put in the points of division of both intervals we shall have the combined interval divided into subintervals of the desired kind. Also, the proposition is true at each point, for by taking $\epsilon_0 < \epsilon/2$ the oscillation in I_0 is less than ϵ, and I_0 requires no further division. The proposition then holds for I.

The uniform continuity of the continuous function follows from the preceding result. Take δ to be the length of the shortest of the subintervals into which I is divided. Then for any x_0 in I we have $|f(x)-f(x_0)| < 2\epsilon$ for $|x-x_0| < \delta$.

4. Sets of points. Let S be a set of points lying on a line. To say that an interval on the line contains no point of S is interval-additive, to say that it contains one point of S, or exactly n points $(n>0)$, is not.

The following corollary of our main theorem is useful here in the statement of results. Its proof is immediate.

THEOREM. *If P is an interval-additive proposition which is not true for a closed interval I, then there is at least one point ξ of I such that P is not true for any subinterval of I which contains ξ.*

The statement that an interval contains a finite number of points of S is interval-additive. Hence, if I contains an infinite number of points of S we have the Bolzano-Weierstrass theorem that I contains at least one *limit point* ξ. In every neighborhood of ξ there are infinitely many points of S.

Also interval-additive, is the statement that an interval contains a finite or denumerably infinite number of points of S. Hence, if there is a non-denumerable set of points of S in I there is a so-called *point of condensation*. In every neighborhood of the point there is a non-denumerable infinity of points of S.

5. Generalizations. Extensions of these ideas to two or more dimensions will occur to the reader at once. A proposition P will be *region-additive* for example, if, whenever P is true for each of two overlapping regions, it is true also for the union of these regions. In a closed bounded base region R we can say that P is *true at a point*, if it is true in some subregion R_0 enclosing the point, where some suitable convention is made to cover the case of points on the boundary.

We can then prove a theorem analogous to our basic theorem and use it to treat a variety of propositions. Properties of functions in two or more variables akin to those discussed in Section 3 and theorems about point sets in R are thus readily established.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1251. *Proposed by J. S. Lew, Princeton University*

"Did your teacher give you that problem?" I asked. "It looks rather tedious."

"No," said Willie, "I made it up. It's a polynomial with my age as a root. That is, x stands for my age last birthday."

"Well, then," I remarked, "It shouldn't be so hard to work out—integer coefficients, integral root. Suppose I try $x=7$ · · · . No, that gives 77."

"Do I look only seven years old?" demanded Willie.

"Well, let me try a larger integer · · · . No, that gives 85, not zero."

"Oh, stop kidding!" said Willie, looking over my shoulder. "You know I'm older than that!"

How old is Willie?

E 1252. *Proposed by Jerome Rothstein, Signal Corps Engineering Laboratories, Fort Monmouth, N. J.*

Prove that the difference between two positive integral powers of the same integer is exactly divisible by six unless the integer gives the remainder two on division by three and one power is odd while the other is even. Show that in this exceptional situation the sum of the two powers is exactly divisible by six.

E 1253. *Proposed by T. R. Jenkins, University of Idaho, and by Nathaniel Macon and Abraham Spitzbart, Flight Propulsion Laboratory, General Electric Co.*

For n and p positive integers with $n>p$, and x and y arbitrary, show that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x+ky)^p=0.$$

Find the value of the sum for $n=p$.

E 1254. *Proposed by Robin Robinson, Dartmouth College*

Prove that if two conics intersect in four distinct points, these points are concyclic if and only if the axes of the two conics are parallel or perpendicular.

E 1255. *Proposed by J. P. Ballantine, University of Washington*

A constant gravitational field operates in the direction of the negative y-axis.

A particle starts from point (a, b) with a given initial velocity, travels along a straight line to the point (x, y), thence without loss of velocity along another straight line to point (c, d). The point (x, y) is so chosen that the total time is a minimum. Prove or disprove that the particle reaches the point (x, y) when the time is precisely half gone.

SOLUTIONS

De Polignac's Theorem

E 1221 [1956, 420]. *Proposed by S. W. Golomb, University of Oslo, Norway*

If n is an integer greater than 1, then 2^n-1 is never a perfect square, cube, or higher power.

Solution by D. C. B. Marsh, Colorado School of Mines. For $n>1$, $2^n-1\equiv 3 \pmod 4$; so, if we were able to write $2^n-1=x^m$, $m>1$, x and m would both have to be odd. But this would imply the impossible relation

$$2^n = x^m + 1 = (x+1)[(x^m+1)/(x+1)],$$

where the final factor is integral, odd, and greater than 1.

Also solved by D. S. Adorno, H. W. Becker, H. F. Bennett, David Bloom, Julian Braun, Leonard Carlitz, P. L. Chessin, M. P. Drazin, L. R. Ford, Jr., George Franzisko, Marshall Freimer and A. L. Tritter (jointly), A. M. Glicksman, I. L. Goldhirsh, R. K. Guy, Virginia S. Hanly, A. R. Hyde, Lawrence Isenecker, Ronald Jacobowitz, J. B. Johnston, M. S. Klamkin and D. J. Newman (jointly), Andrew Kraus, Sidney Kravitz, Joe Lipman, Marshall Luban, E. W. Marchand, Patricia G. Mässé, J. B. Muskat, E. A. Nordhaus, Margaret Olmsted, Anatol Rapaport, Azriel Rosenfeld, C. M. Sandwick, Sr., J. P. Scholz, J. R. Smart, Mirko Stojaković, D. R. Sudborough, A. V. Sylwester, Lincoln Teng, R. M. Warten, R. E. Wyllys, F. H. Young, and the proposer.

Becker called attention to Dickson's *History of the Theory of Numbers*, vol. II, p. 753, where it is pointed out that in 1887 C. de Polignac proved that $a^n-2^k=\pm 1$ is impossible unless $a=3$, $n=1$ or 2. Chessin remarked that the given problem is covered by the solution to E 444 [1941, 482]. Drazin established the stronger theorem: *the equation $p^n-c^m=1$ has no solution in integers c, m, n, p, with p a positive prime and $m>1$, $n>1$, other than that given by $3^2-2^3=1$.*

The Regular Heptagon

E 1222 [1956, 421]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If we designate by C_1, C_2, C_3 the sides of the regular convex heptagon and of the two regular star heptagons inscribed in a circle of radius R, then

$$C_1^2 + C_2^2 + C_3^2 = 7R^2.$$

I. *Solution by P. L. Chessin, Westinghouse Electric Corporation.* We generalize. Let $A_0, A_1, \cdots, A_{n-1}$ be the consecutive vertices of a regular n-gon inscribed in a circle of radius R and center O, and let $C_k=A_0A_k$, $1\leqq k\leqq n-1$.

THEOREM. $\sum_{k=1}^{n} C_k^2=2R^2[m-\sin m\pi/n \cos (m+1)\pi/n \csc \pi/n]$, $1\leqq m\leqq n-1$.

In triangle OA_0A_k apply the law of cosines to obtain $C_k^2=2R^2(1-\cos 2k\pi/n)$. Then

$$\sum_{k=1}^{m} C_k^2 = 2R^2\left[m - \sum_{k=1}^{m} \cos 2k\pi/n\right].$$

But, by a well known result,

$$\sum_{k=1}^{m} \cos k\phi = \sin m\phi/2 \cos (m+1)\phi/2 \csc \phi/2.$$

COROLLARY 1.

$$\sum_{k=1}^{n-1} C_k^2 = 2nR^2.$$

COROLLARY 2. *If n is even,*

$$\sum_{k=1}^{n/2-1} C_k^2 = (n-2)R^2.$$

COROLLARY 3. *If n is odd,*

$$\sum_{k=1}^{(n-1)/2} C_k^2 = nR^2.$$

The given problem is established by Corollary 3 with $n=7$.

II. *Solution by W. B. Carver, Cornell University.* Let $t=e^{2\pi i/7}$. Then t^s, $s=0, 1, \cdots, 6$, are the seven seventh roots of unity, t^{7-s} is the conjugate of t^s, and

$$1 + t + t^2 + t^3 + t^4 + t^5 + t^6 = 0.$$

The vertices of the regular heptagon correspond to the complex numbers Rt^s, and we have

$$C_1^2 = R^2(t-1)(t^6-1), \quad C_2^2 = R^2(t^2-1)(t^5-1), \quad C_3^2 = R^2(t^3-1)(t^4-1),$$

and

$$C_1^2 + C_2^2 + C_3^2 = R^2(3t^7 - t - t^2 - t^3 - t^4 - t^5 - t^6 + 3) = (3+1+3)R^2 = 7R^2.$$

III. *Solution by Norman Anning, Alhambra, Calif.* Put unit masses at the vertices of the regular heptagon. Taking moments of inertia with respect to any vertex and applying the familiar parallel-axis theorem, we have

$$2(C_1^2) + 2(C_2^2) + 2(C_3^2) = 7R^2 + 7R_2^2.$$

Also solved by Leon Bankoff, A. D. Bradley, Leonard Carlitz, Edward Fleisher, Michael Goldberg, Cornelius Groenewoud, R. K. Guy, A. R. Hyde, V. F. Ivanoff, M. S. Klamkin and D. J. Newman (jointly), J. D. E. Konhauser, Sidney Kravitz, Josef Langr, D. C. B. Marsh, Margaret

Olmsted, D. K. Pease, L. A. Ringenberg, C. M. Sandwick, Sr., B. M. Stewart, Chih-yi Wang, F. H. Young, and the proposer.

Bankoff supplied the following items of interest relative to the figure of the problem. In the triangle ABC whose sides are $a \equiv C_1$, $b \equiv C_2$, $c \equiv C_3$, we have:

(1) $bc = a(b+c)$, $ac = b(c-a)$, $ab = c(b-a)$

(2) $\cos A \cos B \cos C = -1/8$, $\sin A \sin B \sin C = \sqrt{7}/8$

(3) $\sin^2 A = (3a-c)/4a$, $\sin^2 B = (3b-a)/4b$, $\sin^2 C = (3c+b)/4c$

(4) $\cos^2 A + \cos^2 B + \cos^2 C = 5/4$, $\sin^2 A + \sin^2 B + \sin^2 C = 7/4$

(5) $\cot V = \cot A + \cot B + \cot C = \sqrt{7}$,

where V is the Brocard angle of triangle ABC

(6) $\cos^2 B \cos^2 C + \cos^2 C \cos^2 A + \cos^2 A \cos^2 B = 3/8$

(7) $h_a = h_b + h_c$,

where h_i are the altitudes of triangle ABC

(8) $h_a^2 + h_b^2 + h_c^2 = (a^2+b^2+c^2)/2$

(9) $b^2/a^2 + c^2/b^2 + a^2/c^2 = 5$

(10) $OH = OI_a = R\sqrt{2}$, $r_a = R/2$, $I_aH = R$

References: *Mathesis*: 1913—204; 1938—169; 1950—344; 1955—329; 1956—106 and 149. *Journal de mathématiques élémentaires*, 69° année, p. 25 (1944).

Editorial Note. Solutions II and III can be extended readily to yield Corollary 3 of Solution I. An easy additional corollary of Solution I is: *In a regular n-gon the sum of the squares of all sides and diagonals is n^2R^2.* An alternative proof of this, using the theory of mean position, may be found in M'Clelland, *A Treatise on the Geometry of the Circle* (1891), p. 103.

Limit of a Curious Sequence

E 1223 [1956, 421]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Starting with two arbitrary non-negative numbers a_1 and a_2, build a sequence a_k, $k = 1, 2, 3, \cdots$, by taking alternately the arithmetic and the geometric means of the two preceding numbers, for example,

$$0,\ 1,\ 1/2,\ \sqrt{2}/2,\ (1+\sqrt{2})/4, \cdots.$$

Evaluate $\lim a_k$ as $k \to \infty$.

Solution by the proposer. If $a_1 = a_2$, then, trivially, $\lim a_k = a_1$. Suppose $a_1 < a_2$. We make use of the following elementary facts: (a) the arithmetic mean of $\cot 2\alpha$ and $\csc 2\alpha$ is $(\cot \alpha)/2$, (b) the geometric mean of $\csc 2\alpha$ and $(\cot \alpha)/2$ is $(\csc \alpha)/2$. Consider the sequence

$$c \cot \theta,\quad c \csc \theta,\quad (c/2) \cot (\theta/2),\quad (c/2) \csc (\theta/2),\quad (c/4) \cot (\theta/4), \cdots,$$

which already fits the desired pattern of alternating arithmetic and geometric means. This can be identified with the desired sequence by choosing c and θ such that $a_1 = c \cot \theta$ and $a_2 = c \csc \theta$, whence $\theta = \cos^{-1}(a_1/a_2)$ and $c = (a_2^2 - a_1^2)^{1/2}$. Since

$$\lim_{n\to\infty} 2^{-n} \cot (\theta/2^n) = \lim_{n\to\infty} 2^{-n} \csc (\theta/2^n) = 1/\theta,$$

the desired limit is

$$\lim a_k = c/\theta = (a_2^2 - a_1^2)^{1/2}/\cos^{-1}(a_1/a_2).$$

For $a_1 > a_2$ the above argument needs but little alteration. If $a_2 = 0$, then $\lim a_k = 0$. If $a_2 > 0$, we can make use of the facts: (a) the arithmetic mean of coth 2α and csch 2α is (coth α)/2, (b) the geometric mean of csch 2α and (coth α)/2 is (csch α)/2, and set

$$a_1 = c \coth u, \qquad a_2 = c \operatorname{csch} u,$$

so that

$$\lim a_k = (a_1^2 - a_2^2)^{1/2}/\cosh^{-1}(a_1/a_2).$$

Also solved by N. J. Fine, M. S. Klamkin, H. L. Krall, D. C. B. Marsh, K. W. Morris, and Chih-yi Wang.

Morris pointed out that the problem appears, although in somewhat different form, in Knopp, *Theory and Application of Infinite Series*, 1928, Ex. 8, parts (c) and (d), p. 41, and Ex. 91, p. 228. Klamkin credited the result to Borchardt, and called attention to Bromwich, *Introduction to the Theory of Infinite Series*, Problem 8, p. 23, and Gibson, *Advanced Calculus*, Problem 3, p. 50.

A Fixed Point Theorem

E 1224 [1956, 421]. *Proposed by G. K. Wenceslas, Providence, R. I.*

Let G be a finite group of rigid motions (or more generally affine transformations) of a Euclidean space. Then there is a point of the space which is left fixed by all the transformations of G.

Solution by N. J. Fine, University of Pennsylvania. Let the group elements T_i be given by $T_i(x) = A_i x + b_i$ $(i = 1, \cdots, n)$, where the A_i are linear. Let c be an arbitrary vector, $x = (1/n)\sum_{j=1}^{n} T_j(c)$. Then

$$T_i(x) = A_i x + b_i = A_i\left[(1/n)\sum_{j=1}^{n} T_j(c)\right] + b_i$$

$$= (1/n)\sum_{j=1}^{n}\left[A_i T_j(c) + b_i\right] = (1/n)\sum_{j=1}^{n} T_i[T_j(c)]$$

$$= (1/n)\sum_{k=1}^{n} T_k(c) = x.$$

Hence x is invariant under the entire group.

Also solved by the proposer.

Editorial Note. One cannot help but notice the following property. Let c and d be any two points and let c_i and d_i be the maps of c and d, respectively, under the transformation T_i. Then, if there is only one fixed point of G, the centroids of the two sets of points $c_1, \cdots, c_n$ and $d_1, \cdots, d_n$ coincide. Examples, like the octic group of the symmetries of a square, are abundant.

Laying a Rug

E 1225 [1956, 421]. *Proposed by L. R. Ford, Illinois Institute of Technology*

Find necessary and sufficient conditions that it be possible to lay an $a \times b$ rug on an $A \times B$ floor. (Take $a \geqq b$, $A \geqq B$.)

Solution by W. B. Carver, Cornell University. The conditions $a \geqq b > 0$ and $A \geqq B > 0$ are imposed by the problem itself and are understood to hold throughout the discussion. There are two cases to be considered: the trivial case when the rug can be placed with its length parallel to the length of the room, and the case when the length of the rug is greater than the length of the floor so that it must be placed diagonally.

In the first case the necessary and sufficient conditions are obviously

$$A \geqq a \text{ and } B \geqq b. \tag{1}$$

In the second case we have $a > A$. In Figure 1 the radius of the circle is

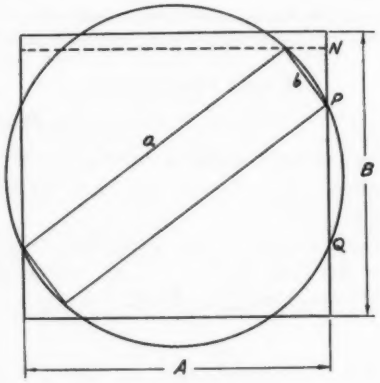


FIG. 1

$(a^2+b^2)^{1/2}/2$, and hence distance $PQ=(a^2+b^2-A^2)^{1/2}$. Letting $PN=h$, we must have

$$B \geqq (a^2 + b^2 - A^2)^{1/2} + 2h. \tag{2}$$

From similar triangles we have

$$h/b = [A - (b^2 - h^2)^{1/2}]/a,$$

and this gives

$$h = [abA - b^2(a^2 + b^2 - A^2)^{1/2}]/(a^2 + b^2).$$

(The larger value of h: $h=[abA+b^2(a^2+b^2-A^2)^{1/2}]/(a^2+b^2)$, satisfies the equation $h/b=[A+(b^2-h^2)^{1/2}]/a$, which has no significance in the problem.) Putting the value of h in (2) we have

$$B \geqq [2abA + (a^2 - b^2)(a^2 + b^2 - A^2)^{1/2}]/(a^2 + b^2). \tag{3}$$

Hence the necessary and sufficient conditions in this case are

$$a > A \text{ and } B \geqq [2abA + (a^2 - b^2)(a^2 + b^2 - A^2)^{1/2}]/(a^2 + b^2). \tag{4}$$

From conditions (4) certain other necessary (but not sufficient) conditions for this second case may be derived, namely, the rather obvious conditions $A^2+B^2 > a^2+b^2$ and $B>b$, and the not so obvious condition $a+b \leqq A\sqrt{2}$.

The necessary and sufficient conditions required in the problem are *either* (1) *or* (4), in the sense that if the rug can be placed on the floor either (1) or (4) must hold, and if either (1) or (4) are satisfied the rug can be placed on the floor.

The inequality conditions (1) and (4) may be represented graphically by means of a rectangular Cartesian system, (a, b, B), in space (see Figure 2).

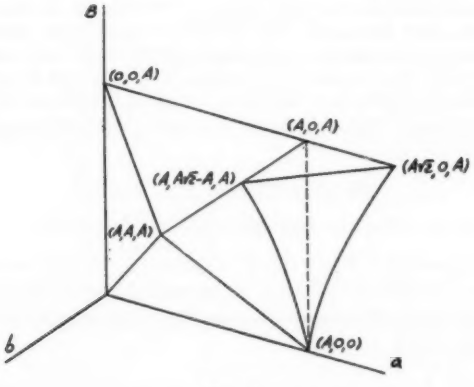


FIG. 2

One plots the points $(0, 0, A)$, $(A, 0, A)$, $(A, 0, 0)$, (A, A, A), $(A\sqrt{2}, 0, A)$, $(A, A\sqrt{2}-A, A)$. Conditions (1) are satisfied by all points in a region which is a quadrangular pyramid with a square base in the plane $b=0$ and vertex at (A, A, A); and conditions (4) by points in the pseudo-tetrahedron bounded by the three planes $a=A$, $B=A$, $b=0$, and a portion of the curved surface

$$B = [2abA + (a^2 - b^2)(a^2 + b^2 - A^2)^{1/2}]/(a^2 + b^2).$$

This triangular piece of curved surface is bounded by the line $a+b=A\sqrt{2}$ in the plane $B=A$, the hyperbola $a^2-B^2=A^2$ in the plane $b=0$, and the curve $(b^2+A^2)B=3A^2b-b^3$ in the plane $a=A$. All points on the boundaries of the regions belong to the regions except points in the planes $a=0$ and $b=0$, and the

points of the plane $a=A$ bounding the second region.

Also solved by H. F. Bennett, D. A. Breault, Michael Goldberg, R. K. Guy, D. C. B. Marsh, C. S. Ogilvy, C. D. Olds, M. J. Pascual, and Azriel Rosenfeld.

A number of the above solutions gave necessary but not sufficient conditions. Guy obtained the necessary and sufficient conditions in the form

$$B \geqq b \text{ and either } A \geqq 0 \text{ or } 3\pi/2 - 2\sin^{-1} a/(a^2+b^2)^{1/2} \leqq \sin^{-1} A/(a^2+b^2)^{1/2} + \sin^{-1} B/(a^2+b^2)^{1/2}.$$

As an example of the second case he gave $a=410$, $b=41$, $A=409$, $B=130$.

Editorial Note. For allied problems see Problem 416 (algebra) [1920, 327], Problem 3036 [1925, 47], and Problem No. 563, *National Mathematics Magazine*, 1945, p. 259.

The necessary condition $a+b \leqq A\sqrt{2}$ is apparent in Figure 2; a purely algebraic derivation of the condition constitutes a nice problem.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4723. *Proposed by Joachim Lambek, McGill University*

Given a non-negative integer n and a prime p, obtain an expression for the number of binomal coefficients $\binom{n}{r}$ which are not divisible by p. (This generalizes problem I, 7 of the Putnam Competition Examination for 1956, this MONTHLY, vol. 64, 1957, p. 24.)

4724. *Proposed by Paul Erdös, Technion Mathematics Department, Haifa, Israel*

Show that the necessary and sufficient condition that

$$\limsup_{x\to 1} (x^{n_1} - x^{n_2} + x^{n_3} - \cdots) = 1$$

is that to every c and ϵ there should exist j and k, $k \geqq j$, such that $n_{2k+2} > cn_{2j+1}$ and

$$(n_{2j+2} - n_{2j+1}) + \cdots + (n_{2k+2} - n_{2k+1}) > (1-\epsilon)(n_{2k+2} - n_{2j+1}).$$

4725. *Proposed by Olga Taussky, National Bureau of Standards*

A square $n \times n$ matrix $A = (a_{ik})$ with complex numbers as elements is called

normal if $AA^*=A^*A$, where A^* is the transposed and complex conjugate of A. A matrix A is called nilpotent if for some integer $r\geqq 1$ the matrix A^r is the zero matrix 0. Prove (by rational methods only) that a normal and nilpotent matrix is the zero matrix.

4726. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If the parallels to the asymptotes of a conic (C), drawn through an arbitrary point P of its plane, intersect (C) in P_1 and P_2, if the perpendiculars to PP_1 and PP_2 at P_1 and P_2 intersect in a point O, and if the polar of P with respect to (C) intersects the conic in M_1' and M_2', then the perpendicular bisector of segment $M_1'M_2'$ passes through O.

4727. *Proposed by G. R. MacLane, the Rice Institute*

Find a function $f(x)$, upper semi-continuous and non-negative on $[0, \infty)$, bounded on each finite interval $(0, T)$, such that $\int_0^\infty f(x)dx=\infty$ and $\sum_{n=1}^\infty f(nh)<\infty$ for every $h>0$. (Cf. problem 4670 [1956, 47, 190]; solution in this number.)

SOLUTIONS

A Sum of Fractional Parts

4667 [1955, 734]. *Proposed by J. L. Ullman, University of Michigan*

Let λ_i be an infinite sequence of positive numbers such that $\sum\lambda_i=1$. Prove that

$$\lim_{N\to\infty}\sum_{i=1}^{\infty}((N\lambda_i))=0,$$

where $((x))$ means the fractional part of x.

Solution by D. S. Greenstein, University of Pennsylvania. Since $0\leqq((x))<1$ and $((x))\leqq x$, we have

$$\frac{((N\lambda_i))}{N}<\frac{1}{N},\qquad 0\leqq\frac{((N\lambda_i))}{N}\leqq\frac{N\lambda_i}{N}=\lambda_i.$$

Given $\epsilon>0$, choose k such that $\sum_{i=k+1}^\infty\lambda_i<\epsilon/2$. Then

$$\left|\frac{1}{N}\sum_{i=1}^{\infty}((N\lambda_i))\right|<\frac{K}{N}+\frac{\epsilon}{2}<\epsilon$$

for sufficiently large N. Therefore the limit is zero as asserted.

Note that no use has been made of the hypothesis $\sum\lambda_i=1$. Hence the conclusion holds if $\sum\lambda_i$ is merely convergent.

Also solved by Leonard Carlitz, A. E. Danese, R. O. Davies, Leopold Flatto, J. B. Kelly, P. G. Kirmser, A. E. Livingston, O. Mourmaki, Chih-yi Wang, J. G. Wendel, and the proposer.

Special Summation

4668 [1956, 46]. *Proposed by D. C. Russell, University of London, England*

Construct the matrix A of a regular method of summation which sums $\sum_{n=0}^{\infty} z^n$ to $(1-z)^{-1}$ at each of the points $z=-1, -2, -3, \cdots$, but which sums the series at no other point in $|z|>1$.

Solution by Albert Wilansky, Lehigh University. Let $v_n^{(m)}$ $(n=0, 1, 2, \cdots, m, \cdots; m=1, 2, 3, \cdots)$ be defined by the equations

$$\sum_{n=0}^{\infty} v_n^{(m)} z^n = \left(\sum_n u_n^{(1)} z^n\right)\left(\sum_n u_n^{(2)} z^n\right)\cdots\left(\sum_{n=0}^{\infty} u_n^{(m)} z^n\right), \tag{1}$$

$$\sum_{n=0}^{\infty} u_n^{(m)} z^n = \frac{m}{m+1} + \frac{z}{m+1}. \tag{2}$$

Let A be the matrix whose mth row is the sequence starting with m zeros and then the sequence $v^{(m)}$, *e.g.*, the second row is

$$(0, 0, v^{(2)}) = (0, 0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{6}, 0, 0, 0, \cdots).$$

A is regular since all its row-sums are one, as is seen by setting $z=1$ in (1) and (2), its elements are non-negative, and its columns terminate in zeros. We now show that if $|z|>1$, A sums the sequence $\{z^n\}$ if and only if z is a negative integer, and to the sum 0 in each case. Since $\sum_{k=0}^{m} z^k=(1-z^{m+1})/(1-z)$, this yields the result.

Let $\{\tau_n\}$ be the transform of $\sum_{n=0}^{\infty} z^n$ by the matrix A. Then

$$\tau_m = \frac{1}{1-z} - \frac{z}{1-z} t_m,$$

where

$$t_m = \sum_{n=0}^{\infty} v_n^{(m)} z^{n+m} = z^m \prod_{r=1}^{m} \frac{r+z}{r+1},$$

as we saw above. If z is a negative integer, $t_m=0$ for sufficiently large m. It remains only to show that $\{t_m\}$ is divergent if $|z|>1$, z not a negative integer. It will be sufficient to show that it is unbounded. The product term is, in absolute value, not less than

$$C \prod_{r=p+1}^{m+1} \left(1 - \frac{w}{r}\right)$$

where

$$w = |z-1|, \qquad C = \left|\prod_{r=1}^{p} \frac{r+z}{r+1}\right| \neq 0,$$

and $p=[w]+1$. This in turn is not less than

$$C\prod_{r=p+1}^{m+1}\left(1-\frac{p}{r}\right)=C\Big/\binom{m+1}{p}\sim Km^{-p}$$

as $m\to\infty$, $K\neq 0$. Thus $|t_m|$ is asymptotically not less than $K|z|^m m^{-p}\to\infty$.

Also solved by Chih-yi Wang and the proposer.

Rectangular Pentagon

4669 [1956, 47]. *Proposed by Michael Goldberg, Washington, D. C.*

1. What is the relation connecting the lengths of the sides of a rectangular skew (spatial) pentagon?

2. Among all possible rectangular skew pentagons, what are the lengths of the sides of the pentagon in which the ratio of the longest to the shortest side is a minimum?

Solution (part 1) by the proposer. Let the lengths of the sides be a, b, c, d, e in that order. Take the plane P of c, d as the xy-plane. If the projection of b on P is x and the projection of e on P is y, while the end of b is below P a distance z_1, and the end of e is above P a distance z_2, then the following five equations are clearly true:

$$z_1^2=b^2-x^2,\qquad a^2+e^2=(d-x)^2+z_1^2+c^2,$$
$$z_2^2=e^2-y^2,\qquad a^2+b^2=(c-y)^2+z_2^2+d^2,$$

$$a^2=(d-x)^2+(c-y)^2+(z_1+z_2)^2.$$

Hence a necessary condition is obtained by the elimination of x, y, z_1, z_2, which gives

$$\{(s^2-a^2-e^2)^2-b^2d^2\}\{(s^2-a^2-b^2)^2-c^2e^2\}=c^2d^2(s^2-c^2-d^2)^2$$

in which we have put $2s^2=a^2+b^2+c^2+d^2+e^2$. Note, as a check, that the relation is unaltered by cyclic permutation of the sides.

Editorial Note. The condition is not sufficient because of squaring during the elimination; *e.g.*, there is no pentagon with sides 1, 1, 1, 1, $\sqrt{6}$. No best almost-equilateral pentagon has been suggested, but E. D. Schell notes that if space is four-dimensional an equilateral rectangular pentagon is possible. For example, consider

$$(0,0,0,1),\quad (0,0,0,0),\quad (1,0,0,0),$$
$$(1,\sqrt{3/8},\sqrt{3/8},1/2),\quad (1/2,(2+\sqrt{5})/2\sqrt{6},(2-\sqrt{5})/2\sqrt{6},1).$$

Divergent integral and series

4670 [1956; 47, 190]. *Proposed by K. L. Chung, Syracuse University*

If $f(x)$ is continuous and non-negative in $[0,\ \infty)$, and $\int_0^\infty f(x)dx=\infty$, then there exists an $h>0$ such that $\sum_{n=1}^\infty f(nh)=\infty$.

I. *Solution by A. R. Hyde, West Hartford, Conn.* Let $k>1$ and

$$M(b)=\frac{1}{k}\int_0^b f(x)dx.$$

$M>0$ for some finite value of b, since $f(x)\geqq 0$ and $M\to\infty$ as $b\to\infty$. Then, if $N\Delta x=b$, we have

$$\lim_{\Delta x\to 0, N\to\infty} \sum_{n=1}^{N} f(n\Delta x)\Delta x = kM,$$

i.e., for some finite positive $\Delta x=h$ and $N=b/h$,

$$\sum_{n=1}^{b/h} f(nh)h > M, \qquad h \lim_{b\to\infty} \sum_{n=1}^{b/h} f(nh) \geqq \lim_{b\to\infty} M,$$

whence the desired result follows.

II. *Solution by N. J. Fine, University of Pennsylvania.* Define the extended real-valued function $G(x)=\sum_{n=1}^{\infty} f(nx)$. It is easy to see that G is lower semi-continuous, *i.e.*,

$$\liminf_{x\to a} G(x) \geqq G(a)$$

for every a. Hence the set $A_N=\{x \mid G(x)\leqq N\}$ is closed.

Now suppose that A_N contains an interval (a, b), $a<b$. Then

$$\int_a^b G(x)dx < \infty.$$

But

$$\int_a^b G(x)dx = \sum_{n=1}^{\infty}\int_a^b f(nx)dx = \sum_{n=1}^{\infty} \frac{1}{n}\int_{na}^{nb} f(x)dx = \int_0^{\infty} \phi(x)f(x)dx,$$

where

$$\phi(x) = \sum_{na<x<nb} \frac{1}{n} = \sum_{(x/b)<n<(x/a)} \frac{1}{n}.$$

As $x\to\infty$, $\phi(x)\to\log(b/a)$, whence $\int_a^b G(x)dx=\infty$. This contradiction shows that A_N contains no interval and, being closed, is nowhere dense. Therefore $A=\{x \mid G(x)<\infty\}=\bigcup_{N=1}^{\infty} A_N$ is of first category, and its complement in $(0, \infty)$ is of second category, hence everywhere dense in $(0, \infty)$.

This has as a corollary Problem 4605 [1955, 738]. For, if S is any unbounded open set, we can construct a non-negative continuous function f such that $\int_0^{\infty} f(x)dx=\infty$ and $f(x)=0$ for $x\notin S$. For this function, $G(x)=\infty$ implies that $nx\notin S$ for infinitely many n.

Also solved by A. E. Danese, A. S. Davis, G. Lorentz, G. R. MacLane, O. E. Stanaitis, and G. N. Wollan.

A Generalization of Taylor's Expansion

4671 [1956; 47, 191]. *Proposed by J.P. Ballantine, University of Washington, Seattle*

Show that, if $f(x)$ is any function possessing $2n+1$ continuous derivatives,

$$\sum_{r=0}^{n} \binom{2n-r}{n} \frac{a^r}{r!} [(-D)^r f(a) - D^r f(0)] = (-1)^n \frac{D^{2n+1} f(\xi)}{(2n+1)!} a^{2n+1},$$

where the first parenthesis under the summation is a binomial coefficient, and ξ lies between 0 and a.

Editorial Note. C. A. Dyche points out that the proposed formula is a special case ($m=n$, $x=0$) of A Generalization of Taylor's Expansion, by Hummel and Seebeck (this MONTHLY, vol. 56, 1949, pp. 243–247). Andress extended the theory to more than two points by means of contour integration (this MONTHLY, vol. 60, 1953, pp. 394–396).

F. B. Hildebrand refers to his treatise, *Introduction to Numerical Analysis*, equations (6.14.6) and (6.14.7). The present result is equivalent to Obrechkoff's quadrature formula with an error term.

Independent solutions were submitted by A. E. Danese, C. A. Dyche, Berthold Schweizer, and Chih-yi Wang.

Integers Uniquely Expressible in Terms of a Sequence

4672 [1956, 47]. *Proposed by D. J. Newman, AVCO Research Division, Lawrence, Mass.*

Find a sequence of positive integers such that almost all (in the usual sense of limit density) positive integers are uniquely expressible as a sum of two of them.

Solution by Paul Erdös. Technion Mathematics Department, Haifa, Israel. Let S_1 be the set of integers $\sum_k a_k 10^{2k}$, $0 \leqq a_k \leqq 9$, and S_2 be the set $\sum_k b_k 10^{2k+1}$, $0 \leqq b_k \leqq 9$. Clearly every integer can be written uniquely in the form $x+y$ where $x \in S_1$ and $y \in S_2$. The numbers x_1+x_2, x_1 and x_2 in S_1, are of the form $\sum a_k 10^{2k} + \sum b_k 10^{2k+1}$ where $0 \leqq a_k \leqq 9$, $0 \leqq b_k \leqq 1$, whence their number up to x is $o(x)$. The same holds for the numbers of the form y_1+y_2, with $y_1, y_2 \in S_2$. Thus the numbers $S_1 \cup S_2$ satisfy the requirements of the problem.

Also solved by the proposer.

A Diophantine Equation

4674 [1956, 126]. *Proposed by R. Venkatachalam Iyer, Karamana, Trivandrum, India*

The general solution in rational numbers of the equation

$$x^2 + y^2 + z^2 + 2xyz = 1, \tag{1}$$

is given by I. A. Barnett as

$$x = \frac{b^2 + c^2 - a^2}{2bc}, \qquad y = \frac{c^2 + a^2 - b^2}{2ca}, \qquad z = \frac{a^2 + b^2 - c^2}{2ab}.$$

(See: *A Diophantine Equation Characterizing the Law of Cosines*, this MONTHLY, vol. 62, 1955, p. 251.) Find the general solution of (1) in integers.

Solution by D. C. B. Marsh, Colorado School of Mines. (1) may be written in the form

$$(x + yz)^2 = (y^2 - 1)(z^2 - 1),$$

which yields solutions if and only if

$$y^2 - 1 = pq^2 \quad \text{and} \quad z^2 - 1 = pr^2,$$

for p, q, r positive integers and p square-free. For p fixed and >1, the Pell equation $s^2 - pt^2 = 1$ has infinitely many solutions. If the least positive integral solution is (s_1, t_1) then the general integral solution is (s_v, t_v) where

$$s_v + t_v\sqrt{p} = \pm (s_1 + t_1\sqrt{p})^v.$$

Hence y and z may be taken as s_m and s_n (both positive or both negative) for any integral m and n, whereupon $x = -s_{m\pm n}$, since

$$x = -s_m s_n \pm pqr = -s_m s_n \pm pt_m t_n.$$

Also solved by Norman Anning, Leonard Carlitz, Walter Penney, Chih-yi Wang, and the proposer.

Editorial Note. Anning traces the problem back to Francis van Schotten the Younger, about 1660. See note, this MONTHLY, 1926, p. 212, also *Enciclopedia della Matematiche Elementari*, vol. I, part 2, pp. 330, 377–8.

Carlitz remarks that a slightly more general equation was solved by P. Bachmann, *Arch. Math. und Physik* (3), vol. 24, 1916, p. 89. See also a paper by L. J. Mordell, *J. London Math. Soc.*, vol. 28, 1953, pp. 500–510.

Infinite Product

4675 [1956, 126]. *Proposed by D. H. Lehmer, University of California at Berkeley*

Show that

$$\left(\frac{3\cdot 3}{1\cdot 5}\right)^4\left(\frac{15\cdot 15}{13\cdot 17}\right)^4\left(\frac{27\cdot 27}{25\cdot 29}\right)^4\left(\frac{39\cdot 39}{37\cdot 41}\right)^4\cdots = 12.$$

Solution by D. C. Russell, University College of North Staffordshire, England. We will employ three well known results concerning the gamma function. (See Whittaker and Watson, *Modern Analysis*, 4th ed., pp. 238–240.)

$$(1) \qquad \prod_{n=1}^{\infty} \frac{(n - a_1)(n - a_2)\cdots(n - a_k)}{(n - b_1)(n - b_2)\cdots(n - b_k)} = \prod_{m=1}^{k} \frac{\Gamma(1 - b_m)}{\Gamma(1 - a_m)},$$

provided that $a_1 + \cdots + a_k = b_1 + \cdots + b_k$,

$$(2)\qquad \Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\cdots\Gamma\left(x+\frac{n-1}{n}\right)=(2\pi)^{(n-1)/2}n^{(1-2nx)/2}\Gamma(nx),$$

$$(3)\qquad \Gamma(x)\cdot\Gamma(1-x)=\pi/\sin\pi x.$$

By (1), the desired product, P, without the fourth power, reduces to

$$P=\Gamma(1/12)\Gamma(5/12)\{\Gamma(1/4)\}^{-2}.$$

(2) and (3) give

$$\Gamma(1/12)\Gamma(5/12)\Gamma(3/4)=2\pi\cdot 3^{1/4}\cdot\Gamma(1/4),\qquad \Gamma(1/4)\Gamma(3/4)=\pi\cdot 2^{1/2},$$

whence $P=2^{1/2}\cdot 3^{1/4}$ which is equivalent to the required result.

Also solved by Leonard Carlitz, A. E. Danese, N. J. Fine, Emil Grosswald, J. R. Hatcher, Peter Henrici, Edgar Karst, M. S. Klamkin, A. E. Landry, A. E. Livingston, Kovina Milosevich, Hermann Schmidt, M. R. Spiegel, Ernst Trost, Chih-yi Wang, J. V. Whittaker, and David Zeitlin.

Arithmetic and Harmonic Means

4676 [1956, 126]. *Proposed by J. E. Wilkins, Jr., Nuclear Development Associates, White Plains, N. Y.*

Let $f(x)$ be a real valued measurable function defined on a measurable set A with measure μ such that

$$0<a\leqq f(x)\leqq b<\infty,$$

almost everywhere on A. Let $\bar{f}$ be the average value of f and $\tilde{f}$ be the reciprocal of the average value of the reciprocal of f. Find least upper and greatest lower bounds for $\bar{f}/\tilde{f}$.

Solution by J. Horváth, University of the Andes, Bogota, Colombia. We have

$$1\leqq\frac{\bar{f}}{\tilde{f}}=\frac{1}{\mu^2}\left(\int_A f\,dx\right)\left(\int_A f^{-1}dx\right)\leqq\frac{1}{4}\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)^2.$$

The bound on the left is attained when f is constant, the bound on the right when $f(x)=a$ on a set B with measure $\mu/2$ and $f(x)=b$ on $A-B$.

The left-hand inequality is the well known one between the arithmetic and the harmonic means (Hardy-Littlewood-Polya, *Inequalities*, p. 144). The one on the right follows from the inequality (*op. cit.*, p. 166, ex. 230)

$$\int_A f_1^2dx\cdot\int_A f_2^2dx\leqq\frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2\left(\int_A f_1f_2dx\right)^2,$$

where $0<m_1\leqq f_1(x)\leqq M_1$, $0<m_2\leqq f_2(x)\leqq M_2$. It suffices in fact to put

$$f=f_1^2,\qquad 1/f=f_2^2,\ m_1=\sqrt{a},\qquad M_1=\sqrt{b},\qquad m_2=\sqrt{1/b},\qquad M_2=\sqrt{1/a}.$$

Also solved by Neill McShane, Chih-yi Wang, Albert Wilansky, and the proposer.

Convergent Series

4677 [1956, 126]. *Proposed by M. S. Klamkin, AVCO Research Division, Lawrence, Mass.*

For what values of θ does the series

$$\sum_{n=1}^{\infty} \frac{1}{n \sin 2^n\theta}$$

converge?

Solution by Leonard Carlitz, Duke University. Put $\theta=\alpha\pi$, where we may assume that $0<\alpha<2$. Also put $\alpha=2^{-n_1}+2^{-n_2}+2^{-n_3}+\cdots$ $(0\leqq n_1<n_2<\cdots)$ and $k_r=n_{r+1}-n_r (r=1, 2, \cdots)$.

Clearly we may assume that $\alpha\neq m/2^t$ where m is an integer. We shall prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n \sin 2^n\alpha\pi} \tag{1}$$

converges if and only if, as $r\to\infty$,

$$2^{k_r}/n_r \to 0, \tag{a}$$

and the following series converges

$$\sum_{r=1}^{\infty} 2^{k_r}/n_r^2. \tag{b}$$

Proof. The necessity of (*a*) is obvious, for otherwise the n_rth term of (1) does not approach zero. In the next place it follows from the identity

$$\frac{1}{\sin 2x}+\frac{1}{\sin 4x}+\cdots+\frac{1}{\sin 2^n x}=\cot x-\cot 2^n x,$$

that

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{n \sin 2^n\alpha\pi} &= \sum_{n=1}^{N} \frac{1}{n}(\cot 2^{n-1}\alpha\pi-\cot 2^n\alpha\pi)\\ &= \cot\alpha\pi-\sum_{n=1}^{N-1}\frac{\cot 2^n\alpha\pi}{n(n+1)}-\frac{1}{N}\cot 2^N\alpha\pi.\end{aligned} \tag{2}$$

It follows from (a) that as $N\to\infty$, $(\cot 2^N\alpha\pi)/N\to 0$. As for

$$\sum_{n=1}^{N-1}\frac{\cot 2^n\alpha\pi}{n(n+1)}, \tag{3}$$

note first that the only negative terms are those for which $n=n_r-1$; since the series

$$\sum_{n=1}^{\infty} \frac{\cot 2^{n_r-1}\alpha\pi}{n_r(n_r-1)}$$

is evidently convergent, we may ignore such terms in (3). In other words, if

$$\sum_{n=1}^{\infty} \frac{\cot 2^{n}\alpha\pi}{n(n+1)} \tag{4}$$

converges, it converges absolutely. Consequently the convergence of (4) implies the convergence of

$$\sum_{r=1}^{\infty} \frac{\cot 2^{n_r}\alpha\pi}{n_r(n_r+1)}; \tag{5}$$

Since the fractional part of $2^{n_r}\alpha$ is equal to $2^{n_r-n_{r+1}}+2^{n_r-n_{r+2}}+\cdots$, it is clear that (5) converges if and only if

$$\sum_{r=1}^{\infty} \frac{2^{k_r}}{n_r(n_r+1)}$$

converges; this is equivalent to (b).

Conversely when (a) and (b) hold, it is clear from (2) that it is only necessary to prove the convergence of (4). But

$$\left|\sum_{n=1}^{N} \frac{\cot 2^{n}\alpha\pi}{n(n+1)}\right| \leqq \sum_{n=1}^{N} \frac{|\csc 2^{n}\alpha\pi|}{n(n+1)},$$

and $\sin 2^{n}\alpha\pi$ is negative only for $n=n_r$. Then the convergence of

$$\sum_{r=1}^{\infty} \frac{\csc 2^{n_r}\alpha\pi}{n_r(n_r+1)}$$

is a consequence of (b), while the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)\csc 2^{n}\alpha\pi}$$

is easily proved by summation by parts as in (2). This completes the proof.

Remark. The series (1) is certainly convergent if the differences k_r are bounded; in particular, (1) converges for rational α (except $m/2^t$). An example of divergence is furnished by $\sum 2^{-n!}$; the condition (a) is not satisfied.

For $n_r=[r\log_2 r]$ both (a) and (b) are satisfied; for

$$n_r = [r(\log_2 r + \log_2\log_2 r)].$$

neither (a) nor (b) holds; while (a) is satisfied but (b) is not if

$$n_r = [r(\log_2 r + \log_2\log_2 r - \log_2\log_2\log_2 r)].$$

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Introduction to Mathematical Logic, vol. I. By Alonzo Church, Princeton University Press, 1956. ix+376 pages. $7.50.

The main body of this book is devoted to a precise and detailed exposition of the propositional calculus and the functional calculi of first and second order. Although intended primarily as a text for a beginning course, the book is capable of serving in other ways. Its use as a standard reference is assured. Perhaps its greatest value will be as a text (in conjunction with vol. II) for those often neglected students who are neither beginners nor experts. Exercises, which are given at the end of most of the sections, increase greatly the value of the book. Both the number and range of the exercises are large. Some of the exercises are routine, but many are extensions or variations of the theory, and a few are unsolved problems.

In broad outline, the book follows the author's 1944 monograph, *Introduction to Mathematical Logic, Part I*. Revisions have been made and much new material has been added. (The monograph has 118 pages and no exercises.) Some of the more important changes and additions are given below.

New is a sixty-eight page introduction, in which the author brings together a number of topics, such as *Names*, *The Logistic Method*, *Syntax*, *Semantics*, which another writer might prefer to scatter throughout the book or include in appendices. Although the author's treatment of this difficult material is uniformly excellent, its massing at the beginning of the book makes it take on the nature of an obstacle. The principal formulation of the propositional calculus is different, and leads quickly to an elegant Kalmár-type proof of deductive completeness. For the pure functional calculus of first order, Henkin's proof of completeness is given in addition to Gödel's. Henkin's method is used again in a completeness proof for the second order calculus. A new section, *Postulate Theory*, discusses systems with non-logical axioms.

The book is not intended to be comprehensive (*e.g.*, modal logic is omitted), but its wide scope is indicated by the presence of forty-one formulations of propositional calculi and thirty-five formulations of functional calculi.

An extensive bibliography is given in the form of footnotes. Two interesting sections are devoted to historical notes. An excellent numbering scheme makes an individual theorem, section or exercise easy to find.

The author has taken great pains to combine precision with clarity, and has succeeded notably. Because the book presupposes "some substantial mathematical background," it is not universally suitable as a text for a first course.

Only a few typographical errors were noted. One worth mentioning occurs in the formula at the top of page 97. (The formula should agree with the preceding one.)

This book will play an important part in the training of mathematical logicians.

ANGELO MARGARIS
Oberlin College

Numerical Analysis. By Z. Kopal. New York, John Wiley and Sons Inc., 1955, xiv+556 pages. $12.00.

This book is intended to be a research handbook as well as a text in numerical analysis as applied to functions of a single real variable. Although elementary calculus and "some algebra" are stated to be the only really necessary prerequisites, nevertheless, it would appear that considerably more mathematical background would be needed in order to understand some of the more advanced chapters.

The book will probably find wider use as a research handbook than as a text for a course in numerical analysis. It would only be possible to cover a fraction of the material in a single semester, and in a two-semester course in numerical analysis, one would almost certainly wish to include a considerable number of topics which are not treated in the book such as algebraic and transcendental equations involving a single variable, systems of equations, matrix problems, and partial differential equations.

The book is written from a point of view considerably removed from that of the user of high speed computing machines and lying somewhere between applied mathematics and hand computation. Some material is included which appears to fall in the area of classical analysis rather than numerical analysis. For instance, four pages are devoted to the standard proof of the Picard iteration process for solving ordinary differential equations and ten pages are used to treat the method of Frobenius. Since these topics are not treated specifically from the point of view of numerical computation, it would seem that only the results could be given, together with a reference to a book on ordinary differential equations.

The introductory chapter provides an interesting history of number systems and of numerical analysis as well as motivations for the study of the subject. Chapter II contains descriptions of the usual methods of polynomial interpolation, including the "throwback" method of Cromie which uses modified differences, and describes an interesting application to curve fitting.

The next chapter contains descriptions of the standard methods for solving ordinary differential equations. In addition, some new material on "successive extrapolation" is developed and applied to equations of the form $y''+f(x)y=g(x)$. The use of adjoint systems to estimate the propagated errors of various numerical procedures is also discussed.

Chapter V which is entitled "Boundary Value Problems" is primarily concerned with characteristic value problems associated with linear ordinary differential equations. By the use of finite difference methods the problem is reduced to that of finding the characteristic values of a matrix. The use of higher order difference equations and also various "extrapolations to zero grid size" are also considered. Chapter VI treats these problems by variational methods including the Rayleigh-Ritz method, Schwarz's method, a "collocation" method which is related to Lagrange's method of interpolation, and a least squares method.

Chapter VII deals with a number of methods for performing numerical quadrature. Methods which use unequally spaced intervals as well as those using equally spaced intervals are discussed. The concluding chapter deals with integral equations and integro-differential equations. The following subjects are treated in the five appendices: operational approach to finite difference formulas; trigonometric interpolation and Tchebysheff polynomials; coefficients for "mechanical" quadrature formulas; and algebraic equations and systems of linear equations.

In general, there seems to be too much attention devoted to some of the topics with the result that the scope is much narrower than one would expect from a book of this length. The book is very clearly written in a pleasing style and the arguments are easy to follow. The chapters on variational methods for solving characteristic value problems and on integral equations are particularly interesting.

At the end of each chapter a fair number of examples are provided together with supplementary notes and some research problems. The book is not entirely devoid of misprints; for instance on page 20, line 27, $a_1, a_2, \cdots, a_n$, should be $a_0, a_1, a_2, \cdots, a_n$ and on page 21 formula (II-B-4) should have $p_n^I(a_j)$ in the denominator.

Although this book is not recommended as a text to be used in a course in numerical analysis, it contains a great deal of interesting material and should prove very useful to those who work in the field of numerical analysis.

DAVID YOUNG
The Ramo-Wooldridge Corporation
Los Angeles, California

Introduction to Numerical Analysis. By F. B. Hildebrand. New York, McGraw-Hill Book Co., Inc., 1956, x+511 pages. $8.50.

In the preface, the author states that the book is intended to provide an introductory treatment of the fundamental processes of numerical analysis which is compatible with the expansion of the field brought about by the development of high speed computing machines. However, he indicates that he intends to take into account the fact that very large amounts of computation will continue to be effected by desk calculators (and by hand or slide rule) and that familiarity with computation on a desk calculator is a desirable preliminary to computation on a high speed computer. He expects that it would be pos-

sible to provide a survey of a substantial portion of the text in a single semester, and that a more thorough coverage could be provided in two semesters.

It is evident that the book will be much more valuable to the user of desk computers than to the user of high speed computers. A better background for the field of modern computation would be provided by a book which covered only the most frequently used methods of hand computation and which at the same time was written from the standpoint of high speed computing. Until such a book appears, however, the present volume will probably find extensive use as a text for courses in numerical analysis. It is very clearly written in sufficient detail so that a good student should be able to learn the material with little or no assistance. At the end of each chapter a large number of illustrative numerical problems are given. These should be of immense value to the instructor, since one of the drawbacks of other available texts in numerical analysis has been the lack of enough good problems.

The introductory chapter contains a discussion of various types of errors, including statistical errors, and analyzes the growth of such errors. Some mathematical preliminaries are also provided. The next four chapters, covering 140 pages, contain a thorough discussion of interpolation, numerical quadrature, and numerical differentiation. Use is made of divided differences, methods based on Lagrange's interpolation formula, finite differences, and difference operators.

The next chapter contains a description of the standard methods for solving ordinary differential equations together with an analysis of so-called "parasitic" solutions and of propagated errors. One section is devoted to two-point boundary value problems and another section describes methods for solving characteristic value problems.

Chapter VII deals with least squares polynomial approximation with special reference to Legendre, Laguerre, Hermite, and Chebyshev approximations. Factorial power functions and summation formulas are also discussed. The chapter is concluded by a description of several techniques for smoothing empirical data. Chapter VIII treats Gaussian quadrature and other related quadrature methods including Hermite, Legendre-Gauss, Laguerre-Gauss, Hermite-Gauss, Chebyshev-Gauss, Jacobi-Gauss, Radan, and Chebyshev quadrature. Chapter IX considers approximations of various types including Fourier approximation for both continuous and discrete ranges, exponential approximation, Chebyshev interpolation, and approximation by continued fractions.

The concluding chapter, which is practically independent of the preceding chapters, summarizes a number of methods for the numerical solution of sets of linear algebraic equations, non-linear algebraic or transcendental equations, and non-linear algebraic equations in particular. There is a bibliography with 276 references and an appendix with a directory of methods.

In general, it seems that too many methods are covered and that there is not enough discrimination nor enough discussion of the relative merits of the various methods. In particular, far too much space is devoted to interpolation and to finite differences. It would certainly seem that a knowledge of only a fraction

of the methods presented would enable one to handle in a satisfactory way all except the most unusual interpolation problems. Except for the section on error formulas, the entire chapter on operator methods might well have been eliminated. Similarly a large number of methods are presented for solving initial value problems for ordinary differential equations and yet only a single section is devoted to the important topic of two-point boundary value problems.

The treatment of methods for solving systems of linear algebraic equations could perhaps have been more detailed, if the length of some of the earlier chapters on interpolation had been reduced.

The material in Chapters VII, VIII, and IX on least squares polynomial approximation, Gaussian quadrature, and approximations, respectively, is very well presented, and does not appear to be readily accessible in as palatable a form elsewhere.

By concentrating only on those methods which are most useful for actual computation, an instructor should be able to present a very satisfactory one semester course in numerical analysis using this book, which is probably the best textbook available today for an elementary course in numerical analysis. The book should be part of the personal library of anyone working in the computing field.

DAVID YOUNG
The Ramo-Wooldridge Corporation
Los Angeles, California

BRIEF MENTION

Publications of potential interest, but which are more properly reviewed in other periodicals, are described briefly below.

Sphere Grid Kit. Sphere Grid Sales, 3829 Davis Place, N.W., Washington 7, D.C. $2.95.

The Sphere Grid is a drafting underlay about 5 inches in diameter of the type discussed in the article *On Spherical Drawing and Computation* by Milton Felstein which appeared in this MONTHLY, vol. 62, 1955, a reprint of which accompanies the Sphere Grid.

Offerings and Enrollments in Science and Mathematics in Public High Schools. By Kenneth E. Brown. Office of Education Pamphlet No. 118. 1956. 24 pages. For sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D.C. 15 cents.

This low cost pamphlet is of considerable interest to persons concerned with the teaching of science and mathematics in the public high schools. This study places emphasis on the *number of pupils* enrolled in schools offering certain courses. It should help disperse the complacency of certain smaller schools that have been lulled into the acceptance of their eviscerated offerings by statistics concerning the *number of schools* offering given courses. For example, about ninety percent of the students in the United States are enrolled in high schools

which offer intermediate algebra, trigonometry, or solid geometry, even though only two-thirds of the high schools offer such courses. Similarly, ninety-four percent of the students in the United States are enrolled in high schools offering either physics or chemistry or both, although only seventy-five percent of the high schools themselves are in this situation. The statistics used in preparing this booklet are from the fall of 1954.

Abacs or Nomograms. By A. Giet, New York, Philosophical Library, 1956. ix +225 pages. $12.00.

A translation by Helen Phippen and J. W. Head of the 1953 publication *Abaques ou Nomogrammes* by A. Giet. It presents nomograms and alignment charts "in an elementary way intended for practical engineers rather than mathematicians." However, many of the actual charts and examples are of sufficient mathematical interest to merit examination by teachers of mathematics.

A Manual of Engineering Geometry and Graphics. By Hollie W. Shupe and Paul E. Machovina. New York, McGraw-Hill Book Co., 1956. vii+347 pages. $5.25.

Chapters on surfaces, map projections, vectors, alignment charts, nomography, and graphic calculus may be of interest to students of mathematics as well as those of engineering geometry for whom the text is primarily intended. The inclusion of a bibliography of visual aids is of interest.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

AAAS CONFERENCE ON MATHEMATICS INSTRUCTION

Recognizing the concern about mathematics instruction and the interest of scientists, as well as mathematicians, in the teaching of mathematics, the AAAS called a Conference on Mathematics Instruction in Washington in October, 1956. The conference was made possible by a grant from the Carnegie Corporation of New York. A report of this conference appeared in *Science*, vol. 124, 1956, p. 1219.

CONFERENCE ON HIGH-SPEED COMPUTERS

The 1957 Conference on High-Speed Computers will be held at Louisiana State University, Baton Rouge, March 5–8, 1957. This conference is open to businessmen, office

managers, accountants, engineers, chemists, physicists, economists, statisticians, and other potential users from all sections of the country. Topics scheduled for discussion by nationally recognized speakers include office procedures, statistical operations and numerical methods designed for the adaptation of problems to machine solution. Several manufacturers of computing equipment will be represented through exhibits or demonstrations of computers in operation.

Inquiries concerning the conference may be directed to C. W. Barnett, Research Programmer, Office of the Comptroller, Louisiana State University, or to Dr. J. W. Brouillette, Director, General Extension Division, Louisiana State University, Baton Rouge 3, Louisiana.

PERSONAL ITEMS

Dean W. L. Duren, Jr., of the University of Virginia represented the Association at a convocation held on November 4, 1956, at Reed College at which the honorary degree of Doctor of Laws was conferred on Dr. F. L. Griffin. Dean Duren gave the principal address, "Mathematics and a Liberal Education." Dr. Griffin has also been honored by the creation of a "Frank Loxley Griffin professorship of mathematics" at Reed College.

Professor W. M. Whyburn of the University of North Carolina was the representative of the Association at the inauguration of President W. T. Gibbs of North Carolina Agricultural and Technical College on November 9, 1956.

Bradley University: Miss Rosamond J. Jones, St. Mary's Seminary Junior College, has been appointed Assistant Professor; Assistant Professor H. E. Sandstrom has been promoted to Associate Professor.

Georgia Institute of Technology: Assistant Professor E. R. Immel, University of Wisconsin, Dr. R. D. Johnson, Jr., University of Virginia, Dr. J. W. Walker, University of North Carolina, and Assistant Professor J. W. Wray, University of Idaho, have been appointed Assistant Professors; Mr. J. W. Jayne and Mr. A. J. Kainen have been appointed Instructors; Assistant Professor B. M. Drucker has been promoted to Associate Professor; Associate Professor A. L. Starrett has been promoted to Professor.

Michigan State University, Department of Statistics: Professor Leo Katz has been named Head of the Department; Professor W. D. Baten, Associate Professors K. J. Arnold, Ingram Olkin, Visiting Associate Professor Gopinath Kallianpur, on leave from the Indian Statistical Institute, Calcutta, Assistant Professors J. F. Hannan, C. H. Kraft A. G. Laurent, Visiting Assistant Professor Morris Skibinsky, on leave from Purdue University, Instructors C. H. Proctor and John Van Dyke are staff members; Professor R. A. Fisher will spend the Fall Quarter, 1957, in the Department as Visiting Distinguished Professor.

Mount Allison University: Mr. J. A. Flemming and Mr. Edgar Sparkes have been appointed Lecturers; Professor W. S. H. Crawford, Head of the Department of Mathematics, has been appointed Dean of Science.

New York University, Department of Mathematics, announces the following transfers: Associate Professor H. E. Wahlert and Assistant Professor John Schoonmaker of Washington Square College to the School of Commerce.

New York University, Institute of Mathematical Sciences: Associate Professor Herbert Greenberg, Carnegie Institute of Technology, has been appointed Associate Professor of Mathematics; Dr. George Witham, Lecturer, University of Manchester, has been appointed Associate Professor of Fluid Dynamics and Applied Mathematics; Dr. Jurgen Moser and Dr. Jerome Berkowitz have been appointed Assistant Professors; Dr. Gian-Carlo Rota, Yale University, has been appointed Research Assistant; Dr. M. Kneser, University of Heidelberg, is a Fulbright Scholar; Dr. Lars Hormander is in residence for the first term of the academic year 1956–57.

Ohio University: Assistant Professors R. K. Butner, W. T. Fishback, and S. J.

Jasper have been promoted to Associate Professors; Dr. W. E. Baxter, University of Pennsylvania, has been appointed Assistant Professor; Miss B. L. Bernhardt, Mrs. N. D. Johnson, Mr. R. N. King, and Miss E. O. Uhl have been appointed Instructors.

Tulane University: Dr. G. B. Preston, Lecturer, Royal Military College of Science, England, has been appointed Visiting Assistant Professor; Dr. H. H. Corson, III, and Assistant Professor V. R. Hancock, Virginia Polytechnic Institute, have been appointed Instructors; Dr. Heinz Renggli, an assistant for higher mathematics at the Swiss Federal Institute of Technology, and Dr. Gert Sabidussi, University of Minnesota, have been appointed Research Instructors; Dr. Anatole Beck, Williams College, has been appointed ONR Research Associate.

University of Colorado: Assistant Professor Arne Magnus, University of Nebraska, and Dr. R. W. McKelvey, a post-doctoral fellow at the University of Maryland, Institute for Fluid Dynamics, have been appointed Assistant Professors; Assistant Professor A. Zirakzadeh, University of Teheran, Iran, has been appointed Instructor; Dr. W. E. Briggs has been promoted to Assistant Professor.

University of Southern California: Dr. Henry Dye, State University of Iowa, has been appointed Associate Professor; Dr. A. V. Balakrishnan, an engineer for the Radio Corporation of America, and Dr. Heinz Cordes, an assistant at the University of Göttingen, have been appointed Assistant Professors; Dr. Douglas Anderson, an associate mathematician at the RAND Corporation, and Dr. John Maybee, University of Minnesota, have been appointed Instructors; Associate Professor A. L. Whiteman has been promoted to Professor.

Vanderbilt University: Professor J. A. Hyden has retired as Head of the Department of Mathematics and is continuing as Professor of Mathematics; Professor E. B. Shanks has been appointed Head of the Department.

Wayne State University: Professor A. L. Nelson has retired as Chairman of the Department and is continuing as Professor of Mathematics; Professor Wallace Givens has been appointed Chairman of the Department.

Wellesley College: Professor Helen G. Russell is on sabbatical leave for the year 1956–57 at Harvard University; Professor R. N. Johanson, Boston University, has been appointed Lecturer for the year.

West Virginia University: Mr. W. L. Anderson, Miss Jean M. Coover, Miss Lois V. Heflin, Mr. W. W. Hokman, Mr. R. P. L. Lu, Mr. B. H. Youell, Jr., and Mr. Harry McClung have been appointed Instructors.

Western Washington College of Education: Dr. H. G. H. Bartram, University of Oregon, has been appointed Instructor; Assistant Professor J. L. Hildebrand has been promoted to Associate Professor.

Mr. Eugene Albert of Brooklyn College is employed as a mathematician at General Electric Company, Schenectady, New York.

Associate Professor R. D. Anderson, University of Pennsylvania, has been appointed Professor at Louisiana State University.

Lt. W. R. Ballard of the Air Force Institute of Technology has been promoted to Assistant Professor.

Associate Professor D. H. Ballou, Middlebury College, has been promoted to Professor.

Mr. R. A. Barron, West Virginia University, has been appointed Assistant Professor at the University of Rhode Island.

Associate Professor S. Louise Beasley of Lindenwood College has been promoted to Professor.

Dr. S. R. Bodner, a senior scientist at AVCO Manufacturing Corporation, Stratford, Connecticut, has been appointed Assistant Professor of Engineering at Brown University.

Assistant Professor Fred Brafman, Wayne State University, has been appointed Assistant Professor at Southern Illinois University.

Mr. J. P. Brannen, an instructor at Sweeny High School, Texas, has been appointed Instructor at Sam Houston State Teachers College.

Dr. N. A. Brigham, a mathematician at the Applied Physics Laboratory, Johns Hopkins University, has a position as a senior scientist at AVCO Manufacturing Corporation, Advanced Development Division, Lawrence, Massachusetts.

Professor Leonard Bristow, Wisconsin State College, has been appointed Professor at the University of Santa Clara.

Assistant Professor C. E. Burgess, University of Utah, has been promoted to Associate Professor.

Dr. J. R. Byrne, Portland State College, has been appointed Assistant Professor at San Jose State College.

Mr. A. A. Caporaso of New York University has accepted a position as a mathematician with the International Business Machines Corporation, New York City.

Mr. M. M. Chirico, a scientist for the Westinghouse Electric Corporation, Atomic Power Division, Pittsburgh, Pennsylvania, has a position as an engineer at the American Machine and Foundry Corporation, Greenwich, Connecticut.

Dr. C. J. Clark of the Continental Oil Company has accepted a position as research scientist with Lockheed Aircraft Corporation, Palo Alto, California.

Miss Virginia Clover, University of Arizona, is employed as an aeronautical research engineer by the National Advisory Committee on Aeronautics, Ames Aeronautical Laboratory, Moffet Field, California.

Mr. W. R. Cooper, a teacher at Northeastern Academy, New York City, has been appointed Assistant Professor at Knoxville College.

Miss Helen E. Core, Northwestern Michigan College, is on leave of absence for the year 1956–57 and is at the University of Hawaii as an instructor.

Mr. R. J. Cormier, Teaching Assistant, University of Tennessee, has been appointed Assistant Professor at Northern Illinois State College.

Mr. C. G. Cullen, Graduate Assistant, University of New Hampshire, has been appointed Instructor at Worcester Polytechnic Institute.

Mr. G. J. Duffy, a research technician at Argonne National Laboratory, Lemont, Illinois, has been appointed an assistant mathematician in the Physics Division of the Laboratory.

Professor H. S. Everett of the University of Chicago has retired.

Miss Constance Foley, West Virginia University, has a position at the University of New Hampshire.

Mr. W. Y. Gateley, Clarkson College of Technology, has been appointed Assistant Professor at Colorado College.

Dr. R. D. Glauz, University of California, Los Alamos Laboratory, New Mexico, has accepted a position as a specialist with the General Electric Company, Cincinnati, Ohio.

Dr. A. J. Goldman, Princeton University, has a position as a mathematician at the National Bureau of Standards, Washington, D.C.

Miss Louisa S. Grinstein, Teaching Fellow, University of Michigan, has been appointed Instructor at Hunter College.

Mr. W. J. Hardell, Michigan State University, is employed as a mathematician at Remington-Rand UNIVAC, St. Paul, Minnesota.

Mr. D. F. Hayes, a student at St. Mary's University, has a position as an engineer with Boeing Airplane Company, Seattle, Washington.

Mr. W. A. Heinly, a statistician in the Office of Special Contracts, University of Pittsburgh, Washington, D.C., is employed as an engineer at the Monroe Calculating Machine Company, Morris Plains, New Jersey.

Associate Professor J. G. Herriot, Stanford University, has been promoted to Professor.

Mr. R. E. Hill, a student at the University of California, has a position as a computer for the Shell Development Company, Emeryville, California.

Mr. David Horwitz, an associate engineer for the Armour Research Foundation, Chicago, Illinois, has a position as a materials engineer at the Englander Company, Chicago, Illinois.

Assistant Professor L. Aileen Hostinsky, Pennsylvania State University, has been promoted to Associate Professor.

Mr. Walter James, a research engineer for Automatic Control Company, St. Paul, Minnesota, has been appointed an associate scientist for the St. Anthony Falls Hydraulics Laboratory, University of Minnesota.

Mr. Douglas Jones, a student at the University of Oklahoma, has a position as an aerophysics engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Associate Professor J. R. F. Kent of Harpur College has been promoted to Professor.

Dr. Leo Lapidus, Michigan State University, has accepted a position as a senior research engineer at Consolidated-Vultee Aircraft Corporation, San Diego, California.

Mr. Martin Lipschutz, Fairleigh Dickinson University, has been awarded a research grant for the academic year.

Mr. J. C. McCully, University of Rhode Island, has been appointed Assistant Professor at Western Michigan College.

Dr. Knox Millsaps, Massachusetts Institute of Technology, has accepted a position as chief scientist with the Holloman Air Development Center, New Mexico.

Professor Stella R. Mizell, Chowan College, has been appointed Instructor at Simpson College.

Mr. Dewey Moore, Naval Ordnance Laboratory, has accepted a position as senior engineer with the Glenn L. Martin Company, Baltimore, Maryland.

Mr. J. T. Morse has been appointed Instructor at Boston University.

Mr. H. A. Mortensen, a student at Utah State Agricultural College, has been appointed Chairman of the Department of Mathematics of Box Elder High School, Brigham City, Utah.

Assistant Professor T. J. Pignani, University of North Carolina, has been appointed Assistant Professor at the University of Kentucky.

Professor R. J. Pitts, Fort Valley State College, has been appointed Assistant Professor at Los Angeles State College.

Dr. R. G. Pohrer of the Chemical Corps, U.S. Army, has been appointed to the staff of the Mathematics Division of the Air Force Office of Scientific Research, Washington, D.C.

Dr. Anthony Ralston, Massachusetts Institute of Technology, has been appointed a member of the technical staff of Bell Telephone Laboratories, Whippany, New Jersey.

Mr. J. G. Renno, Jr., University of Wisconsin, has accepted a position as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. B. E. Rhoades of Lafayette College has been promoted to Assistant Professor.

Mr. S. T. Rio, Pacific University, has been promoted to Assistant Professor.

Mrs. Jane Ingersoll Robertson, University of Maryland, has been appointed Instructor at the University of Illinois.

Mr. D. A. Rux, University of Kansas, has been appointed Professor and Chairman of the Department of Mathematics of Wisconsin State College, Oshkosh.

Associate Professor Judson Sanderson, Jr., U.S. Air Force Institute of Technology, has been appointed Associate Professor at the University of Redlands.

Mr. E. T. Sheffield, a physicst at the U.S. Naval Radiological Defense Laboratory, San Francisco, California, has been appointed Instructor at California State Polytechnic College.

Professor Jack Silber has returned to Roosevelt University after spending the spring and summer semester as Consultant to the Assistant for Operations Analysis, U.S. Air Force.

Mr. W. B. Simmons, Jr., an associate engineer at Lockheed Aircraft Corporation, Burbank, California, is now employed as an engineer at the American Bosch Arma Corporation, Garden City, New York.

Sister Mary Corona, Cardinal Stritch College, has been promoted to Assistant Professor.

Dr. C. V. L. Smith, a scientific liaison officer for the Office of Naval Research, London Branch, is now Chief of the Computation Laboratory, Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Mr. R. J. Smith has been appointed Instructor at Queen's University, Kingston, Ontario, Canada.

Mr. R. F. Steinhart, New Jersey State Teachers College, Montclair, has a position with the International Business Machines Corporation, Newark, New Jersey.

Dr. Leonard Tornheim, University of California, has accepted a position as a research mathematician with the California Research Corporation, Richmond, California.

Miss Eugenia I. Trapp, Tulane University, has a position as applied science representative for the International Business Machines Corporation, Houston, Texas.

Mr. C. H. Tross, Lockheed Aircraft Corporation, has accepted a position as head mathematician of the research laboratory, American Bosch Arma Corporation, Garden City, New York.

Assistant Professor R. Z. Vause, Jr., Vanderbilt University, has been appointed Assistant Professor at the University of Kansas City.

Professor J. A. Ward, University of Kentucky, has a position as a mathematician at Holloman Air Force Base, New Mexico.

Dr. G. P. Weeg, previously employed by the Sperry Rand Corporation, Remington Rand UNIVAC, St. Paul, Minnesota, has been appointed Assistant Professor at Michigan State University.

Dr. Chien Wenjen, Knoxville College, has been appointed Assistant Professor at Texas Technological College.

Mr. R. E. Wild, Teaching Assistant, University of California, Los Angeles, has a position as a computer analyst for Douglas Aircraft Corporation, Santa Monica, California.

Associate Professor A. G. Wootton, State University of New York, has been appointed Lecturer at the University of Maine.

Dr. Fumio Yagi, Ballistic Research Laboratories, Aberdeen Proving Ground, New Mexico, has accepted a position as a senior research engineer with the Jet Propulsion Laboratory, California Institute of Technology.

Assistant Professor F. H. Young, Portland State College, is now with the Autonetics Division of North American Aviation Company, Downey, California.

Professor Witold Hurewicz, Massachusetts Institute of Technology, died on September 6, 1956.

Professor Walter Reynolds, Georgia Institute of Technology, died on September 10, 1956.

Professor Emeritus L. W. Smith, Washington and Lee University, died on August 9, 1956. He was a charter member of the Association.

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The thirty-fifth annual meeting of the National Council of Teachers of Mathematics will be held at the Bellevue-Stratford Hotel, Philadelphia, March 27–30, 1957. Inquiries concerning the meeting may be sent to the National Council, 1201 Sixteenth Street, N. W., Washington 6, D.C.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

ITINERARIES OF VISITING LECTURERS, 1956–57

A. W. Tucker

Clarkson College of Technology	Potsdam, N. Y.	Oct. 1–3
St. Lawrence Univ.	Canton, N. Y.	Oct. 4–5
Georgetown Univ. and Catholic Univ. of America	Washington, D. C.	Oct. 15–20
North Carolina State College	Durham, N. C.	Oct. 22–24
Randolph-Macon Woman's Coll. and Lynchburg Coll.	Lynchburg, Va.	Nov. 12–14
Univ. of Richmond	Richmond Va.	Nov. 15–17
Univ. of Tennessee	Knoxville, Tenn.	Nov. 26–27
Georgia Institute of Technology	Atlanta, Ga.	Nov. 28–Dec. 1
Univ. of Georgia	Athens, Ga.	Nov. 28–Dec. 1
Alabama Polytechnic Institute	Auburn, Ala.	Dec. 3, 4, 5
Univ. of Alabama	Tuscaloosa, Ala.	Dec. 6–8
Univ. of Mississippi	Oxford, Miss.	Dec. 10–11
Univ. of Houston	Houston, Tex.	Dec. 13–14
North Texas State Coll.	Denton, Tex.	Dec. 17–19
Tulane Univ.	New Orleans, La.	Dec. 20–21
Dartmouth Coll.	Hanover, N. H.	Jan. 21–23
Franklin and Marshall Coll.	Lancaster, Pa.	Feb. 11–12
State Teachers Coll. and Wilson Coll.	Shippensburg, Pa. and Chambersburg, Pa.	To be arranged
Vassar Coll.	Poughkeepsie, N. Y.	Feb. 25–27
Drew Univ.	Madison, N. J.	March 4–6
Allegheny Coll.	Meadville, Pa.	April 4–5
Univ. of Buffalo	Buffalo, N. Y.	April 8–11
Kent State Univ.	Kent, Ohio	April 25–27
Miami Univ.	Oxford, Ohio	April 29–30
Univ. of Cincinnati	Cincinnati, Ohio	May 1–2
Xavier Univ.	Cincinnati, Ohio	May 3–4
Purdue Univ. (Fort Wayne Center)	Fort Wayne, Ind.	May 6–7
Valparaiso Univ.	Valparaiso, Ind.	May 8–9
DePauw Univ.	Greencastle, Ind.	May 10–11
Knox Coll.	Galesburg, Ill.	May 13–14
Southern Illinois Univ.	Carbondale, Ill.	May 16–17

Edwin Hewitt

Sacramento State Coll.	Sacramento, Calif.	March 27–28
Fresno State Coll.	Fresno, Calif.	March 28–29
California Institute of Technology	Pasadena, Calif.	April 1–2
Univ. of Redlands	Redlands, Calif.	April 3–5
Arizona State Coll.	Tempe, Ariz.	April 8–10
Univ. of Arizona	Tucson, Ariz.	April 11–12
New Mexico Coll. of Agriculture and Mechanic Arts	State College, N. M.	April 15–17
Univ. of Colorado and environs	Boulder, Colo.	April 22–24
Univ. of Nebraska	Lincoln, Neb.	April 25–26
Univ. of Utah and Utah State Agricultural Coll.	Salt Lake City and Logan, Utah	April 29–May 3
Montana State Univ.	Missoula, Mont.	May 7–9
State Coll. of Washington and Univ. of Idaho	Pullman, Wash. and Moscow, Idaho	May 13–17
Coll. of Idaho	Caldwell, Idaho	May 20–21

Kurt Mahler

N. J. State Teachers Coll.	Montclair, N. J.	March 28
Vanderbilt Univ. and George Peabody Coll. for Teachers	Nashville, Tenn.	April 8–11
Bowling Green State Univ.	Bowling Green, Ohio	April 12–15
Kenyon Coll. and Denison Univ.	Gambier and Granville, Ohio	April 16–19
Emmanuel Missionary Coll.	Berrien Springs, Mich.	April 22–23
Univ. of North Dakota	Grand Forks, N. D.	April 25–26
Concordia Coll.	Moorhead, Minn.	April 29–30
Carleton Coll. and St. Olaf Coll.	Northfield, Minn.	May 1–3
Iowa State Coll., State Univ. of Iowa and Iowa State Teachers Coll.	Ames, Iowa City, and Cedar Falls, Iowa	May 6–10
Central State College	Edmond, Okla.	May 13–14
Univ. of Kansas, Washburn Univ., Municipal Univ. of Wichita, and Bethel Coll.	Lawrence, Topeka, Wichita, and North Newton, Kan.	May 15–24

J. L. Walsh

Canisius Coll.	Buffalo, N. Y.	April 1–2
Univ. of Rochester	Rochester, N. Y.	April 3–5
Univ. of Vermont	Burlington, Vt.	April 8–9
Middlebury Coll.	Middlebury, Vt.	April 10
Skidmore Coll.	Saratoga Springs, N. Y.	April 11
Cardinal Stritch Coll.	Milwaukee, Wis.	April 12
Univ. of Wisconsin (Milwaukee Ext.)	Milwaukee, Wis.	April 15–16
Amherst Coll. and Univ. of Massachusetts	Amherst, Mass.	April 17–19
Wesleyan Univ. and Trinity Coll.	Middletown and Hartford, Conn.	April 22–24
Univ. of Delaware	Newark, Del.	April 26–29
St. John's College and St. John's Univ.	Jamaica, N. Y.	April 30

Continuation of the Program of Visiting Lecturers

The National Science Foundation has granted the Association the sum of $55,200 for the support of a continuation of the Program of Visiting Lecturers for a two year period beginning in September 1957. This is the third grant for this purpose from the National Science Foundation. The present grant will allow the continuation of the program after the conclusion of the period covered by the previous grants.

The Association's Committee on Visiting Lecturers consists of Professors G. B. Huff, R. A. Rosenbaum, D. E. Richmond, and B. W. Jones, Chairman. The Committee is authorized to select lecturers and to arrange their itinerary. Any correspondence on either of these topics should be addressed to Professor Jones as Chairman of the Committee.

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 67 persons have been elected to membership by the Board of Governors on applications duly certified.

C. D. ABBATE, B.A. (Western S.C.) Math., Holloman Air Force Base, N. Mex.

G. C. ANDERSON, B.A. (Concordia) Grad. Asst., University of Nebraska

H. B. ANDERSON, M.A. (Michigan) Asst. Professor, Michigan College of Mining and Technology.

R. S. BACKEN, B.S. (North Dakota Agric. C.) Grad. Asst., North Dakota Agricultural College.

E. A. BEHRENS, Student, University of Washington.

D. J. BOYCE, B.S. (Central S.C.) Grad. Asst., Oklahoma Agricultural and Mechanical College.

D. A. BREAULT, Student, Carnegie Institute of Technology.

J. W. CASON, M.A. (Southern Methodist) Instr., Cuttington College, Suakoko, Liberia.

P. L. DUREN, A.B. (Harvard) Res. Asst., Massachusetts Institute of Technology.

R. S. ELLZEY, Student, University of Mississippi.

J. R. ENO, JR., B.A. (Whittier) Acting Instr., University of Idaho.

L. F. EPSTEIN, Ph.D. (M.I.T.) Res. Asso., Knolls Atomic Power Lab., General Electric Co., Schenectady, N. Y.

VILLA E. FENDER, A.M. (Kansas) Head, Department of Mathematics, College High School, Bartlesville, Okla.

N. H. FISHER, JR., M.A. (California) Math. & Statistical Aid, Ballistics Research Lab., Aberdeen Proving Ground, Md.

L. D. FOUNTAIN, S.M. (Chicago) Grad. Asst., University of Nebraska.

NANCY E. FRENCH, B.A. (Russell Sage) Lab. Asst., Knolls Atomic Power Lab., General Electric Co., Schenectady, N. Y.

K. D. FRYER, Ph.D. (Toronto) Asso. Professor, Royal Military College of Canada.

D. J. GARDA, Student, Lebanon Valley College.

H. R. GILLETTE, B.A. (Reed) Res. Engr., North American Aviation, Los Angeles, Calif.

F. M. C. GOODSPEED, Ph.D. (Cambridge, England) Asso. Professor, University of British Columbia.

L. A. HART, B.S. (Loras) Instr., Loras College.

PATRICIA HAUSS, A.B. (Indiana) Math. Teacher, Clinton High School & Junior College, Iowa.

ROY HEATH, B.A. (Omaha) Grad. Asst., University of Nebraska.

G. F. HECK, II, Student, Lebanon Valley College.

D. M. HESS, Student, Fordham University.

G. A. HEUER, M.A. (Nebraska) Instr., Concordia College.

RAYMOND HIRSCHKOP, B.S. (Rutgers) Instr., Pratt Institute.

FLORENCE E. INGHAM, M.A. (Teachers C., Columbia U.) Math. Teacher, College High School, Bartlesville, Okla.

I. M. ISAACS, Student, Polytechnic Institute of Brooklyn.

E. D. JACOBSON, B.S. (Columbia) Instr., State University of New York, Agricultural and Technical Institute, Alfred.

M. S. JOHNSON, B.A. (N.Y.U.) Math., Melpar, Falls Church, Va.

C. D. KEIM, Student, Gannon College.

R. F. KELLER, B.S. in Ed. (Southeast Missouri S.C.) Instr., School of Mines and Metallurgy, University of Missouri.

G. H. KIEL, Student, Polytechnic Institute of Brooklyn.

REV. R. F. KING, O.F.M., B.S. (Siena) Asst. Professor, Siena College.

MARTIN LEVEY, Ph.D. (Dropsie C.) Instr., Temple University.

J. S. LINNEKIN, Student, Lebanon Valley College.

R. S. LOCKHART, M.A. (Teachers C., Columbia U.) Head, Department of Mathematics, Madison High School, N. J.

G. G. LORENTZ, Ph.D. (Tübingen, Germany) Professor, Wayne State University.

F. J. LORENZEN, JR., S.B. (M.I.T.) Grad. Asst., University of New Hampshire.

J. B. LOVE, B.A. (Pennsylvania) Instr., Lebanon Valley College.

K. B. MACMURRAUGH, Student, Stanford University.

HELEN M. MCPHERSON, Student, University of Mississippi.

E. J. MILLER, B.S. (Calif. I.T.) Comptroller's Staff, C. F. Braun & Co., Alhambra, Calif.

H. W. MOORE, M.A. (Missouri) Numerical Analyst, General Electric Co., Cincinnati, Ohio.

H. E. MORRISON, B.S. (Johns Hopkins) Electronics Components Design Engr., Glenn L. Martin Co., Baltimore, Md.

D. M. OLSON, B.A. (Dana) Grad. Asst., Oklahoma Agricultural and Mechanical College.

R. K. OTNES, B.A. (Nebraska) Grad. Asst., University of Nebraska.

DON PASSMAN, Student, Polytechnic Institute of Brooklyn.

W. H. PEIRCE, Ph.D. (Wisconsin) Asst. Professor, Michigan State University.

R. S. PINKHAM, Ph.D. (Harvard) Res. Asst., Princeton University.

H. O. POLLAK, Ph.D. (Harvard) Res. Math., Bell Telephone Labs., Murray Hill, N. J.

V. R. RAIL, M.S. (S.U. of Iowa) Asst. Professor, Parsons College.

FRANK RAUNIKAR, B.S. (Oklahoma) Grad. Asst., Oklahoma Agricultural and Mechanical College.

WALTER ROTH, M.S. (S.U. of Iowa) Asst. Professor, Southeast Missouri State College.

DIRAN SARAFYAN, Mech. Eng. (Toulouse, France) Asso. Professor, Lamar State College of Technology.

E. G. SCHULD, M.A. (Wisconsin) Instr., University of Wisconsin, Milwaukee.

C. S. SMITH, M.A. (S.U. of Iowa) Asst. Professor, Drury College.

R. F. STEINEN, M.A. (Montclair S.T.C.) Asst. Professor, Montclair State Teachers College.

D. C. TERRELL, M.A. (Arkansas) Asst. Professor, Southern State College.

G. A. THOMAS, Student, Lebanon Valley College.

V. D. TURNER, M.A. (Illinois) Instr., Mankato State Teachers College.

MERLYN VANDERBEEK, B.S. (Nebraska) Grad. Asst., University of Nebraska.

C. K. VILIM, JR., B.S. (Northwestern) Math., Analog Computer Div., Aerial Measurements Lab., Evanston, Ill.

W. C. WALTER, A.B. (New York S.T.C., Albany) Grad. Asst., University of Wisconsin.

C. W. WILLIAMS, Ph.D. (Virginia) Asso. Professor, Washington and Lee University.

M. E. WINGER, M.S. (North Dakota) Instr., University of North Dakota.

THE OCTOBER MEETING OF THE MINNESOTA SECTION

The fall meeting of the Minnesota Section of the Mathematical Association of America was held at Concordia College, Moorhead, Minnesota, on October 6, 1956. Professor

Sigurd Mundhjeld presided at the morning session and Professor Walter Fleming, Chairman of the Section, presided at the afternoon session. The meeting was attended by 44 persons, including 30 members of the Association.

By invitation of the Executive Committee, Mr. G. A. Heuer, Concordia College, delivered an address at the morning session entitled "Invariant Measure on Topological Groups." An abstract of this address follows:

This paper outlines a development of Haar measure on locally compact groups. Neither the results nor the methods are new. The Daniell extension of an elementary integral to a Lebesgue type integral is discussed (*Ann. of Math.*, vol. 19; pp. 279–294). The elementary integral is an additive, homogeneous, non-negative functional which is continuous under the taking of monotone limits, defined on a vector space L of bounded real-valued functions on a set, where L is also closed under the lattice operations. An elementary integral is then obtained on the vector space L of the real-valued continuous functions with compact support on a locally compact group. Daniell's extension is the Haar integral. The paper concludes with the uniqueness theorem for the Haar integral and with the observation that as examples of the Haar integral on the real numbers, the integers and finite discrete groups, the Lebesgue integral, infinite summation and finite summation, respectively, are special cases of a unifying theory.

The following short papers were presented:

1. *A slide rule hint*, by Mr. Louis Van Slyck, North Dakota Agricultural College, introduced by Professor A. G. Hill.

A slide rule operation which is very important in electrical engineering is that of determining the magnitude and direction of a resultant from its components. This can be done in one motion of the slide on any rule which has the tangent and sine scales on the slide and a D scale on the body of the rule, provided the ratio of the smaller to the larger quantity is not less than 0.1. A description of this operation is given in this paper and the exceptional case is discussed.

2. *Imaginary roots from graphs*, by Professor P. A. Rognlie, University of North Dakota.

It was shown that certain properties of the graphs of quadratic and cubic functions can be used to determine the imaginary roots of the corresponding equations.

3. *A model of a cycloidal pendulum*, by Professor E. J. Camp, Macalester College.

A physical model was presented with a cycloidal pendulum and a simple pendulum suspended from the same frame. The cycloidal pendulum was constrained to move along a cycloidal path by suspending it from the point of tangency of the two arcs of the evolute to the path. The periods of the two pendulums were compared for small and large oscillations. A stop watch was used to show that the period of the cycloidal pendulum remained constant while the period of the simple pendulum increased with the amplitude. The differential equation of the path and the formula for the period of each pendulum was developed.

4. *Plane linkage models*, by Mr. M. E. Winger, University of North Dakota.

Plane linkages were described and illustrated by examples of models constructed by the author. The history and uses of linkages were discussed briefly beginning with the problem of producing rectilinear motion. Mentioned and illustrated were Watt's Parallel Motion, the Peaucellier Cell, the Hart Cell and several linkages producing special curves.

5. *Some undergraduates discover "mathematics,"* by Professor J. M. Calloway, Carleton College.

The paper reports the work of three students with little mathematical background on the representation of an integer as the sum of consecutive integers. The interest of this paper is in the

circumstances of the work rather than the result which is: An integer can be represented as the sum of consecutive integers if and only if it is not a power of two, and the number of representations is equal to the number of odd divisors (one excluded).

6. *Use of a high-speed computer in evaluating a probability*, by Dr. David Gosslee, North Dakota Agricultural College, introduced by Professor A. G. Hill.

The author presented a statistical problem which led to experimental sampling using an IBM 650 computer to obtain probabilities related to the distribution of the statistic involved. In one case experimental sampling was more efficient than direct calculating performed with a high-speed computer. In the second case, the use of experimental sampling was necessary since the distribution of the statistic was unknown. The manner in which the sampling experiment was conducted on the computer and some capabilities of the computer were discussed.

F. C. SMITH, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 31, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, May 4, 1957.

ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.

INDIANA, May 4, 1957.

IOWA, Iowa State Teachers College, Cedar Falls, April 26–27, 1957.

KANSAS, University of Kansas, Lawrence, April 13, 1957.

KENTUCKY, Berea College, Berea, April 20, 1957.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1957.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Johns Hopkins University, Baltimore, Maryland, May 4, 1957.

METROPOLITAN NEW YORK, Hunter College, New York, April 27, 1957.

MICHIGAN, Wayne State University, Detroit, March 23, 1957.

MINNESOTA, Carleton College, Northfield, May 11, 1957.

MISSOURI, Southeast Missouri State College, Cape Girardeau, April 27, 1957.

NEBRASKA, University of Nebraska, Lincoln, April 26, 1957.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, University of Cincinnati, April 20, 1957.

OKLAHOMA, University of Arkansas, Fayetteville, April 12–13, 1957.

PACIFIC NORTHWEST, State College of Washington, Pullman, June 14, 1957.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado School of Mines, Golden, May 3–4, 1957.

SOUTHEASTERN, Emory University, Emory University, Georgia, March 15–16, 1957.

SOUTHERN CALIFORNIA, San Diego State College, May 11, 1957.

SOUTHWESTERN, University of Arizona, Tucson, April 26–27, 1957.

TEXAS, University of Houston, Houston, April, 1957.

UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.

WISCONSIN, Wisconsin State College, Whitewater, May 11, 1957.

opportunities for
mathematicians and statisticians
at the

GENERAL ELECTRIC RESEARCH LABORATORY

The Information Studies Section of the General Electric Research Laboratory in Schenectady has several challenging opportunities for mathematicians and statisticians who are prepared for and interested in pursuing a career of fundamental research in such fields as the logic of information processing machines, and information theory.

Prompt attention will be given to inquiries addressed to Dr. Lewis Eldred, Research Laboratory, General Electric Company, Box 1088, Schenectady, New York.

Mathematicians *for Analysis Group of*

Modern Research & Development Laboratory

Principal fields of interest are: Operations research, weapons system evaluation, application of atomic energy. Experience in any of these areas: Probability theory, optimization procedures, exterior ballistics, fire control and navigation systems, electromagnetic theory, acoustics or aerodynamics. Several openings are available at all levels of training and experience in each of the following:

Operations Research • Operations Analysis
Applied Mathematics • Numerical Analysis

These openings require men with vision and initiative. Our modern laboratory provides a professional working atmosphere and the location in a quiet suburban area provides pleasant living and working conditions. Liberal employee benefit program including tuition refund plan.

All inquiries in confidence. Please send resume, including salary desired to Personnel Manager.

Vitro LABORATORIES | 200 Pleasant Valley Way
West Orange, New Jersey
Division of Vitro Corporation of America

THE AMERICAN MATHEMATICAL MONTHLY

Complete volumes from volume 23 (1916) to volume 52 (1945) inclusive, are for sale at the following rates:

Single volumes:	$10 per volume
Five or more volumes (any years):	$ 5 per volume

From volume 53 (1946) to volume 63 (1956), the price per volume is $10.

We pay transportation charges if payment accompanies order.

Send orders to:

Harry M. Gehman, *Secretary-Treasurer*
Mathematical Association of America
University of Buffalo
Buffalo 14, New York

THE ENJOYMENT OF MATHEMATICS

By HANS RADEMACHER AND OTTO TOEPLITZ

This book introduces the reader to some of the fundamental ideas of mathematics, the ideas that make mathematics exciting and interesting. Amateurs will savor this book, and sophisticated mathematicians will enjoy seeing again how mathematics can be both simple and deep.

240 pages. 123 figures. $4.50

CONTRIBUTIONS TO THE THEORY OF NONLINEAR OSCILLATIONS, Volume III

Edited by S. LEFSCHETZ

Contributors to this new volume of studies include George Seifert, Lawrence Markus, Edmund Pinney, V. B. Haas, R. E. Gomory, Samuel Barocio, Felix Haas, George Hufford, S. P. Diliberto, M. D. Marcus, Paul Koosis and W. T. Kyner, *Annals of Mathematics Study No. 36.*

300 pages. $4.00

Order from your bookstore, or

PRINCETON UNIVERSITY PRESS, Princeton, N.J.

COLLEGE GEOMETRY

by Leslie H. Miller

Here is an important new college textbook for courses in advanced Euclidean plane geometry. The book stresses geometric constructions, with more emphasis than most texts on alternate methods of solution, extension of construction methods to general or to related problems, and discussions of special cases. The development of advanced topics centers about geometric transformations with emphasis on inversion. The discussion of inversion is preceded by homothetic figures and followed by an introduction to properties of the projective transformation. The text concludes with a discussion of special problems of a more advanced nature designed to challenge superior students and a brief treatment of some of the classic unsolvable problems of geometry. 117 figures and 414 exercises. *The Appleton-Century Mathematics Series.* To be published in March.

APPLETON-CENTURY-CROFTS, INC.

35 West 32nd Street • New York 1, N.Y.

ELEMENTS OF MATHEMATICS

by J. Houston Banks, George Peabody College for Teachers

Giving your students a worthwhile and significant treatment of beginning Mathematics, this text is designed specifically for freshmen. Written at the student level, it stimulates their interest and curiosity with challenging problems and a fresh approach to traditional material.

Four broad aspects of elementary Mathematics are covered—Number, Proof, Measurement and Function. Detailed explanations are enhanced by a logical development designed to hold the students' interest. Understanding is emphasized, as opposed to mechanical manipulation. And, no prerequisites beyond Arithmetic are needed.

422 pages • 6" x 9" • Published 1956

examination copies available, write to:

ALLYN AND BACON COLLEGE DIVISION

Englewood Cliffs, New Jersey

Ph. D. MATHEMATICIANS

Take part in the design of atomic reactors for naval propulsion at Combustion Engineering's Nuclear Research and Development Center, located on a 535 acre site in the beautiful Connecticut valley near Hartford.

Permanent positions available in the numerical analysis of programming problems for high-speed digital computers. Previous programming experience desirable but not required.

- **LONG ESTABLISHED COMPANY**
- **OPPORTUNITY FOR INDIVIDUAL GROWTH**
- **LIBERAL BENEFITS.**

A NEW DIVISION OF A PIONEER IN THE MANUFACTURE OF STEAM GENERATING EQUIPMENT

Submit Resume to
Frederic A. Wyatt

C-E

COMBUSTION ENGINEERING, INC.

REACTOR DEVELOPMENT DIVISION, WINDSOR, CONN.

RONALD *Textbooks*

SOLID GEOMETRY—Ready in March

Hugo Mandelbaum *and* Samuel Conte—*both Wayne State University*

This new textbook offers a sound introduction to modern mathematical thinking consistently applied to the teaching of elementary solid geometry. The emphasis is more on understanding and application and less on axiomatics and systematic proofs of theorems. Concepts of projective geometry are presented to break the limitations imposed on Euclidean geometry. These concepts allow the student to view cylinders and prisms, cones and pyramids as interrelated to each other by their mode of generation.

The idea of dualism helps the student to conceive manifold details as a whole by a principle that unifies the details on a higher level. The modern concepts presented are used as basic tools to explore and integrate new subject matter throughout without sacrificing the usual material on relationships of points, lines, planes, and the measurements of all the more common solids.

Numerous good-sized and clear illustrative figures, many of them shaded to facilitate space conception, accompany the text, examples, and exercises. The text is divided into self-contained sections with illustrative example problems which are completely solved. These are followed by an ample supply of well graded, stimulating exercises.

294 ills., tables; 235 pp.

ANALYTIC GEOMETRY

Alfred L. Nelson, Karl W. Folley, *and* William M. Borgman
—*all Wayne State University*

Prepared for use in a freshman course in analytic geometry, this textbook is planned as preparation for the calculus, rather than a study of geometry. In order that it may be of maximum value to the future student of the calculus, the basic sciences, and engineering, considerable attention is given to two important problems—given the equation of a locus, draw the curve, or describe it geometrically; given the geometric description of a locus, find its equation. Stresses coordinate system. *"Concise, well written, and sprinkled with stimulating graphs."* **The American Mathematical Monthly.**

114 ills., tables; 215 pp.

TRIGONOMETRY

Roy Dubisch, *Fresno State College*

This popular textbook mirrors a conspicuous trend in the teaching of plane trigonometry by presenting trigonometric functions as functions of real numbers, with trigonometric functions of angles as a supporting topic. This approach relates the subject more closely to other courses in mathematics. An introductory chapter gives a careful definition of functions in general, developing the concept of a function as a rule of correspondence between two sets of objects and emphasizing the distinction between a function and a function value. *"Ideal for the student who intends to take more advanced work in mathematics."* **R. N. Van Arnam, Lehigh University.** With arc length protractor and scale.

119 ills., tables; 396 pp.

SLIDE RULE PROBLEMS

With Operational Instructions

Philip J. Potter, Edward O. Jones, Jr., *and* Floyd S. Smith
—*all Alabama Polytechnic Institute*

JUST PUBLISHED. The major purpose of this new book is to supply a large number of slide rule problems for student solution. The number is sufficient to obviate any need of repeating for several terms. Preceding the problems are nine short text selections on slide rule operations with numerous example problems. The section on dimensional analysis has been included because of its importance to the engineering student. All pages are perforated. Answers to problems available to instructors.

8⅛ x 10⅞. Illus. 191 pp.

THE RONALD PRESS COMPANY • 15 East 26th St., New York 10

Algebra for College Students

REVISED EDITION

Especially effective for students with weak preparation in mathematics

by JACK R. BRITTON *and* L. CLIFTON SNIVELY, *University of Colorado*

The refreshingly clear exposition, attractive appearance and easy-to-understand, step-by-step presentation of subject matter have made this text a national favorite. Teachers in colleges and universities across the country, large and small, have found it particularly helpful in meeting the widespread problem of the mathematically ill-prepared student. 537 pp., $4.50

Calculus

by JACK R. BRITTON

Exceptionally teachable organization. 3200 exercises, problems, and review questions

Based on years of careful planning, class testing, and teaching experience and written by an author widely known for the clarity, accuracy and reliability of his college mathematics texts, this new presentation of the calculus is outstandingly easy to follow and at the same time mathematically rigorous. 584 pp., $6.50

Engineering Mathematics

by KENNETH S. MILLER, *New York University*

Just published

The author's unusually wide experience in current engineering research and development is reflected in the inclusiveness of this thorough modern treatment of differential equations, network theory and random functions. 417 pp., $6.50

Understanding Arithmetic

by ROBERT L. SWAIN, *State University of New York*

Now in press

This lively new approach to fundamental number concepts, with many original illustrations, will be particularly helpful to those preparing to teach in the elementary schools. It not only gives a very thorough, clear understanding of fundamental ideas and relationships, but also presents many interesting ways of making these fundamentals clear to others.

We will be glad to send you examination copies to consider for text use in your classes.

RINEHART
& COMPANY, INC.

232 Madison Avenue **New York 16, N.Y.**

2 Popular Mathematics Texts

A MODERN INTRODUCTION TO MATHEMATICS

by JOHN E. FREUND, Virginia Polytechnic Institute

Features: 1) Designed to meet the modern trend in the teaching of beginning mathematics in colleges and universities, namely, to stress the *logical principles;* 2) a rather novel introduction to probability and statistics, stressing the ideas of statistical inference rather than descriptive statistics; 3) exercises and worked out examples in practically all sections.

543 pages · 6" x 9" · Published 1956 · Text list $6.00

COLLEGE ALGEBRA

by MOSES RICHARDSON, Brooklyn College

This extremely lucid text gives unusual insight into sound mathematics. It is adaptable to classes of any degree of preparation and combines careful explanation of procedure with reasonable motivation for the student. Your students will not find it excessively rigorous—where correct proof would be too difficult, a searching discussion is substituted. The difficult sections have been starred both in the text itself and in the Table of Contents.

472 pages · 6" x 9" · Published 1947 · Text list $4.00

Approval copies available from:

PRENTICE-HALL, INC.

Englewood Cliffs, New Jersey

HEATH College Texts

Analytic Geometry and Calculus

Published this month

WILLIAM L. HART

Designed for students having no formal previous acquaintance with analytic geometry.

Introduces integration early, in a chapter restricted to the case of algebraic functions. Commences with fifteen lessons devoted solely to the most basic analytic geometry.

The more advanced content in plane analytic geometry is woven in with the calculus.

Transformation of coordinates by rotation of axes, treatment of the general equation of the second degree, and other topics of plane analytic geometry not essential in elementary calculus are segregated in an optional chapter.

The first third of the book could provide a three-hour semester course covering basic analytic geometry, plus a substantial part of differential calculus and a foundation chapter on integral calculus, restricted to algebraic functions, with applications.

Relatively early location for the complete content on multiple integrals.

Text: 648 pages. Tables and Appendix: 32 pages

Mathematical Analysis

E. J. CAMP

An integrated treatment of topics from college algebra, trigonometry, analytic geometry, and calculus for the freshman year.

Some distinguishing features include:

- Full, clear explanations
- Introduction of a new concept as a specific problem to be solved
- Derivatives and integrals introduced early and used throughout
- Complex numbers treated as number pairs
- Unusually extensive and thorough treatment of polar coordinates, limits, and determinants
- Inclusion of a review chapter on radicals

561 pages of text. $6.25

D. C. Heath and Company

Sales Offices: Englewood, N.J., Chicago 16, San Francisco 5, Atlanta 3, Dallas 1
Home Office: Boston 16

INTRODUCTORY COLLEGE MATHEMATICS

By Robert W. Wagner, University of Massachusetts. In press

The author aims to give the student a concept of mathematics somewhat closer to a mathematician's view than is usually attained in an introductory course. After introductory chapters on numbers and equations, the *function concept* is carefully presented and becomes the central theme of the book. Emphasis is on the interdependence of various aspects of the traditional content of freshman mathematics, with an effort made to minimize the number of formulas to be memorized. There is a careful and thorough treatment of each basic problem. Numerous problems are included.

INTRODUCTION TO STATISTICAL ANALYSIS

By W. J. Dixon and Frank J. Massey, both of the University of California, Los Angeles. New Second Edition. 488 pages, $6.00

An excellent revision of one of the most popular of mathematical statistics texts. With no calculus prerequisite, it has been adopted in a variety of situations ranging from math departments and business administration, to biology and agriculture. It presents the basic concepts of statistics in a manner which will show the student the generality of the application of the statistical method. Much material has been brought up to date, with the latter chapters expanded to include several important topics.

CALCULUS AND ANALYTIC GEOMETRY

By C. T. Holmes, Bowdoin College. 416 pages, $5.50

Designed for a combination course in which the concepts and techniques of the calculus are the main objectives. Although calculus is emphasized, the essentials of analytic geometry are presented in sufficient detail for a subsequent major. The concept of integration is introduced early in the text.

TRIGONOMETRY

By Cecil T. Holmes, Bowdoin College. 246 pages (with tables), $3.50

Here is a clearly understandable and thorough text that offers a closer correlation of trigonometry with other college mathematics than is done in most books of its kind. This is accomplished by introducing some very elementary notions from analytic geometry and using them consistently in proofs throughout the book. Emphasis is upon understanding, rather than mechanical manipulation.

Send for copies on approval

McGraw-Hill Book Company, Inc.

330 West 42nd Street New York 36, N.Y.

7 texts by Paul R. Rider *Professor Emeritus, Washington University and Statistical Advisor, Wright-Patterson Air Force Base, Dayton, Ohio*

FIRST YEAR MATHEMATICS FOR COLLEGES

This book contains the basic topics for a first-year course in mathematics. Each chapter is developed so that the teacher can adapt the book to his own course requirements.

1949 — 714 pp. — $5.00

INTERMEDIATE ALGEBRA FOR COLLEGES

Designed for students having only one year of high school algebra, this text offers a clear explanation of the fundamentals, with extensive illustrative problems and carefully graded problems.

1949 — 242 pp. — $3.75

COLLEGE ALGEBRA, Revised Edition

Retaining the first edition's clear style, this text features an expanded discussion of basic algebraic concepts, completely new sets of exercises, and an improved method of presenting generalized concepts.

1955 — 352 pp. — $4.00

COLLEGE ALGEBRA, Alternate Edition

This edition offers a different series of selected and graded exercises, combined with much useful material from the original edition.

1947 — 407 pp. — $4.00

PLANE TRIGONOMETRY

Defining trigonometric functions in terms of their right angle identities before the general trigonometric formulas, this text presents a thorough study of plane trigonometry with numerous, graded exercises.

1953 — 180 pp. — $3.15

PLANE AND SPHERICAL TRIGONOMETRY

Here is a complete text adaptable to any order of presentation. The first problems are simplified so that the student can grasp principles and learn methods.

1942	without tables	275 pp.	$3.50
	with tables	418 pp.	$4.00

ANALYTIC GEOMETRY

Without sacrificing sound mathematical rigor, this book provides the student with practical applications of the methods he is studying, including extensive illustrations and problems.

1947 — 383 pp. — $4.00

The Macmillan Company

60 FIFTH AVENUE, NEW YORK 11, N.Y.

Distributors in Canada
BRETT-MACMILLAN LTD.
23 HOLLINGER ROAD, TORONTO 16

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.